MST204.17

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 17
The dynamics of many-particle systems

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Unit 17
The dynamics of many-particle systems

Prepared by the Course Team

The Open University

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First published 1982. New edition 1990. Reprinted 1993, 1995.

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Designed by the Graphic Design Group of the Open University.

Typeset by the TEXproject, Faculty of Mathematics, The Open University.

Printed and bound in the United Kingdom by Staples Printers Rochester Limited, Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT.

ISBN 0 7492 2033 3

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Introduction

So far the mechanical models used in this course have been single-particle systems, where a particle represented the one mobile physical object which was assumed to have non-negligible mass. We turn in this unit to the study of systems involving more than one particle, which are referred to collectively as *many-particle systems.*

In *Unit 4* we invited you to think of the single representative particle as being situated at a point within the object called its *centre of mass.* We claimed that the motion of the centre of mass could be predicted by considering this coincident particle to have the same mass as the object and to be subjected to all of the external forces which act upon the object. This unit will provide some justification for that claim.

Unit 4 Subsection 1.1

The single-particle model works well in many applications, but as noted in *Unit 4* there are cases where it does not contain the essence of the behaviour under consideration. As an obvious instance, the system to be analysed might consist of two or more massive objects capable of distinct but dependent motion (two pucks at either end of a spring, for example). The motions of these objects cannot be described using just one particle, though an adequate description might be obtained with a model which features a particle corresponding to each object. Nor can a particle by itself model the rotational motion of an object, such as the rolling of a ball on a sloping surface. The subject of rotation will be dealt with in *Unit 29.*

A further application for which the single-particle model proves inadequate is that of rocket motion. A rocket accelerates forwards by burning fuel and ejecting the exhaust gases through a nozzle in its tail. Any attempt to explain this effect by modelling the rocket and all of its exhaust gases as a single particle is bound to fail. We need instead a model that consists of at least two particles, one to represent the rocket with its unspent fuel and the other for the expelled gases.

In order to analyse a model which includes several particles, Newton's second law may be applied to each particle in turn. However, it is not just the number of particles which makes this a more complicated operation than that for the single-particle case. In general, each particle may experience forces due to the presence of the other particles, and these *inter-particle forces* may vary in a complicated way or simply be unknown.

If the forces acting between the particles are unknown then application of Newton's second law cannot lead directly to a description of the individual particle trajectories. In order to make progress we assume that the inter-particle forces satisfy *Newton's third law,* which was stated at the beginning of *Unit 4* but has not subsequently been referred to. The third law says that the two forces acting mutually between a pair of objects are equal in magnitude but act in opposite directions. With this assumption (which matches what is observed in practice) it becomes possible to eliminate the inter-particle forces from direct consideration by adding together the equations of motion for the individual particles, and the result is the equation of motion for the system's centre of mass.

If it is not just the centre of mass motion which is of interest then the problem remains of analysing the individual particle motions. An important and tractable case occurs when the inter-particle forces are significant only for a short time interval, which may be thought of as the duration of a *collision.* In this case it is possible to relate the particle motions before and after the collision by applying principles of conservation. One quantity that may be conserved during a collision is the total mechanical energy of the system. Another is the total *momentum,* or resistance to being stopped.

You saw how the conservation of mechanical energy may be used to deduce the motion of a single particle in *Unit 7* Section 3 and in *Unit 15* Subsection 5.4.

In general a system of many particles will require two or three dimensions for its description, so that where Newton's second law is to be applied it is the vector formulation of *Unit 15* Section 2 which is called for. However, it is often helpful to consider a one-dimensional case before generalizing to three dimensions. Also, although several results in the unit apply to all many-particle systems, most of the examples to be investigated involve only two particles.

Study guide

This unit contains five sections, of which Sections 2, 3 and 5 should be studied in order. Section 4 (with television) summarizes the main ideas in Sections 2 and 3. The first section (with audio-tape) may be studied at any time. It investigates a specific type of two-particle system in which the particles are joined by a string passing over a pulley.

Section 2 starts with another example of a two-particle system, where the particles are joined by a perfect spring and move in one dimension. Then Newton's second and third laws are used to establish the general equation of motion for the centre of mass of a many-particle system.

In Section 3 the concept of momentum is introduced, first for a single particle and then for a system of particles. It is shown how the conservation of momentum, and in some cases the conservation of energy, can be applied to collisions between particles.

Section 5 derives a differential equation for the motion of a rocket. The model used here features particles with variable mass, in contrast to the constant-mass particle models considered earlier.

The television programme in Section 4 provides a summary of what is in Sections 2 and 3, though it may usefully be viewed at any stage during your study of the unit.

1 Two-particle string–pulley systems (Audio-tape Section)

In this section we look at some model systems involving two particles joined by a string which passes over a pulley. It is assumed for modelling purposes that the string is weightless, and that the pulley is weightless and frictionless. Under these circumstances the tension in the string has the same magnitude T on either side of the pulley. This can be seen from the symmetry of the diagram of forces acting upon the pulley, which is shown in Figure 1. The force of magnitude R is the reaction exerted upon the pulley by the bearing at its fixed centre. This reaction force is directed along the line bisecting the two parts of the string.

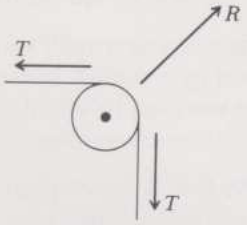
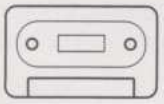


Figure 1

Each particle in these systems is constrained to move along a straight line, so that a starting point for the analysis is to consider them individually as single particles in one-dimensional motion. It is then possible to combine their equations of motion by taking into account the physical connection between the particles.

Start the audio-tape when you are ready.



1 Example 1

particle 1 — pulley

m_1

Assume:

- no friction
- pulley weightless
- string weightless and inextensible
- system initially at rest

m_2 — particle 2

2 Newton's Second Law...

for Particle 1

R, x_1, m_1, T, $m_1 g$

Vertically: $R = m_1 g$

Horizontally:

$$m_1 \ddot{x}_1 = T \quad (1)$$

for Particle 2

T, m_2, $m_2 g$, x_2

Vertically:

$$m_2 \ddot{x}_2 = m_2 g - T \quad (2)$$

3 The Motion of the System

'string inextensible' $\Longrightarrow \dot{x}_1 = \dot{x}_2$

$\Longrightarrow \ddot{x}_1 = \ddot{x}_2 \Longrightarrow m_2 \ddot{x}_1 = m_2 g - T \quad (3)$

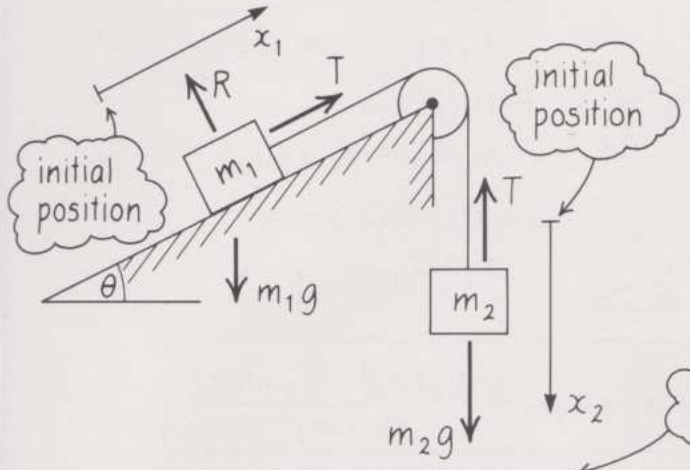
Add Equations (1) & (3): $(m_1 + m_2)\,\ddot{x}_1 = m_2 g. \quad (4)$

So $\ddot{x}_1 = \ddot{x}_2 = m_2 g/(m_1 + m_2)$, ← constant

$\dot{x}_1 = \dot{x}_2 = m_2 g t/(m_1 + m_2) + v_0$, ← 'system initially at rest' $\Longrightarrow v_0 = 0$

$$x_1 = x_2 = \tfrac{1}{2} m_2 g t^2/(m_1 + m_2) + x_0.$$ ← choice of origins for x_1 & $x_2 \Longrightarrow x_0 = 0$

4 Example 2



Assume:

- no friction
- pulley weightless
- string weightless and inextensible
- system initially at rest

Newton's 2nd law

$m_1\ddot{x}_1 =$ ☐ (1) $m_2\ddot{x}_2 =$ ☐ (2)

5 The Motion of the System

Add Equations (1) & (2): $\ddot{x}_1 = \ddot{x}_2$: inextensible string

$$(m_1 + m_2)\,\ddot{x}_1 = m_2 g - m_1 g \sin\theta.$$

So $$\ddot{x}_1 = \ddot{x}_2 = g(m_2 - m_1 \sin\theta)/(m_1 + m_2), \quad (3)$$

(3): constant

$$\dot{x}_1 = \dot{x}_2 = gt(m_2 - m_1 \sin\theta)/(m_1 + m_2),$$

$\dot{x}_1 = \dot{x}_2$: initial velocity zero

$$x_1 = x_2 = \tfrac{1}{2} g t^2 (m_2 - m_1 \sin\theta)/(m_1 + m_2).$$

$x_1 = x_2$: initial position zero

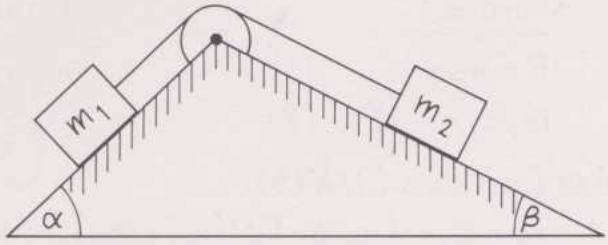
3 cases:

- $m_2 > m_1 \sin\theta$: particle 1 rises, particle 2 descends;
- $m_2 < m_1 \sin\theta$: particle 1 descends, particle 2 rises;
- $m_2 = m_1 \sin\theta$: system remains stationary.

6 Procedure

1. Mark in forces. (for each particle)
2. Choose axis. (for each particle)
3. Apply Newton's 2nd law.
4. Use properties of system to obtain differential equation in one unknown.
5. Solve and/or interpret.

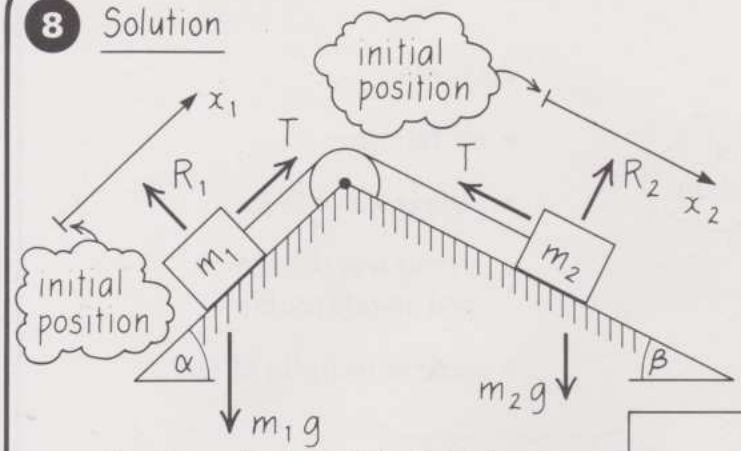
7 Example 3



Assumptions are as in Frame 4.

When does the system (a) move right; (b) move left; (c) remain stationary?

8 Solution



Particle 1:

$$m_1 \ddot{x}_1 = T - m_1 g \sin\alpha. \quad (1)$$

Particle 2:

$$m_2 \ddot{x}_2 = m_2 g \sin\beta - T. \quad (2)$$

Inextensible string:

$$\ddot{x}_1 = \ddot{x}_2.$$

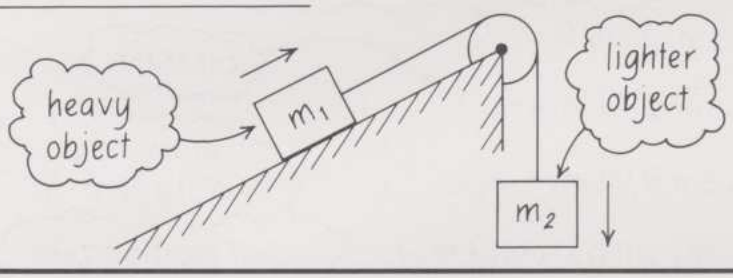
Add Equations (1) & (2):

$$\ddot{x}_1 = \ddot{x}_2 = \frac{g(m_2 \sin\beta - m_1 \sin\alpha)}{m_1 + m_2}$$

System has constant acceleration (initially at rest):

- moves to right if $m_2 \sin\beta > m_1 \sin\alpha$;
- moves to left if $m_2 \sin\beta < m_1 \sin\alpha$;
- remains stationary if $m_2 \sin\beta = m_1 \sin\alpha$.

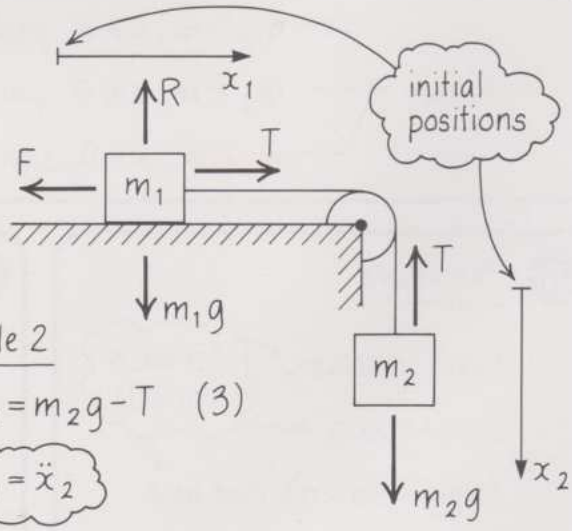
9 An Atwood Machine



'Machine' moves heavy object up the plane, using weight of lighter object.

10 Example 4

Consider system of Frame 1, but with 'no friction' assumption replaced; assume a constant coefficient of friction μ (static and kinetic).



Particle 1

$$R = m_1 g \quad (1)$$

$$m_1 \ddot{x}_1 = T - F \quad (2)$$

Particle 2

$$m_2 \ddot{x}_2 = m_2 g - T \quad (3)$$

$\ddot{x}_1 = \ddot{x}_2$

Add Equations (2) & (3):

$$\ddot{x}_1 = \ddot{x}_2 = (m_2 g - F)/(m_1 + m_2). \quad (4)$$

Case 1: $m_2 \leq \mu m_1$

$F = T = m_2 g$

No motion: $x_1 = x_2 = 0$

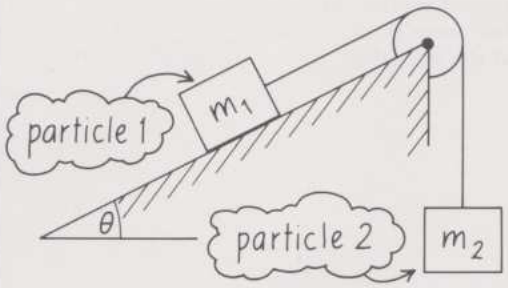
Case 2: $m_2 > \mu m_1$

max F

$F = \mu R = \mu m_1 g$

$\ddot{x}_1 = \ddot{x}_2 = g(m_2 - \mu m_1)/(m_1 + m_2)$

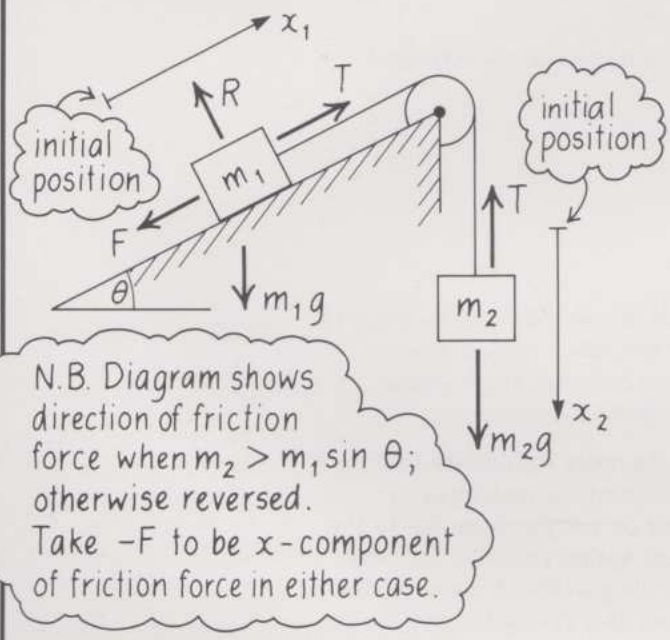
11 Example 5



Consider system of Frame 4, but with 'no friction' assumption replaced; assume a constant coefficient of friction μ (static and kinetic).

When does particle 1 (a) move right; (b) move left; (c) remain stationary ?

12 Solution



N.B. Diagram shows direction of friction force when $m_2 > m_1 \sin\theta$; otherwise reversed. Take $-F$ to be x-component of friction force in either case.

Particle 1

$$R = m_1 g \cos\theta$$

$$m_1 \ddot{x}_1 = T - F - m_1 g \sin\theta \qquad (1)$$

Particle 2

$$m_2 \ddot{x}_2 = m_2 g - T \qquad (2)$$

Add Equations (1) & (2):

$$\ddot{x}_1 = \ddot{x}_2 = \frac{g(m_2 - m_1 \sin\theta) - F}{m_1 + m_2}$$

$|F| \leqslant \mu R = \mu m_1 g \cos\theta.$

Motion of particle 1:

- moves to right if $m_2 > m_1(\sin\theta + \mu\cos\theta)$;
- moves to left if $m_2 < m_1(\sin\theta - \mu\cos\theta)$;
- remains stationary if $m_1(\sin\theta - \mu\cos\theta) \leqslant m_2 \leqslant m_1(\sin\theta + \mu\cos\theta)$.

13 Summary

1. Mark in forces.
2. Choose axis.
3. Apply Newton's 2nd law.
4. Use properties of system to obtain differential equation in one unknown.
5. Solve and/or interpret.

for each particle

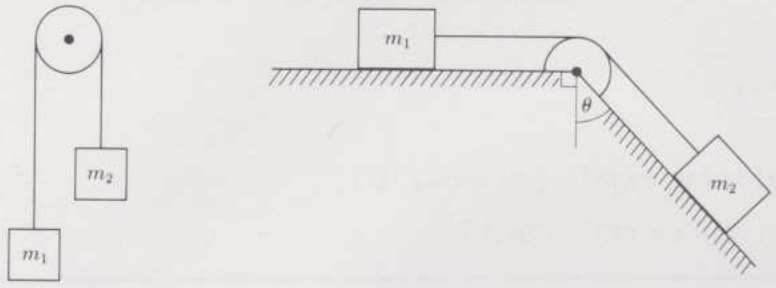
e.g. 'inextensible string'

Procedure for two-particle system

Now try the following exercises, using the procedure in Frame 13.

Exercise 1

For each of the systems of particles shown in Figure 2 below, find the equations of motion for the particles, assuming that the the strings and pulleys are weightless and that there is no friction. Solve the equations of motion to find expressions for the positions of the particles, assuming that each particle is initially at rest.



(i) (ii)

Figure 2

Exercise 2

Modify your solution to Exercise 1(ii) for the case where there is friction on the horizontal surface but not on the inclined plane.

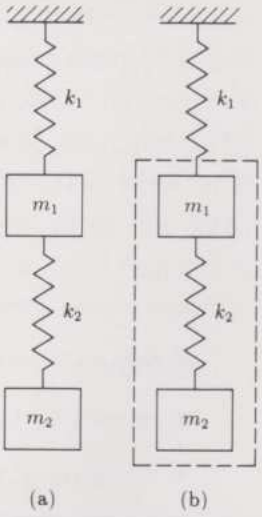
[*Solutions on page 48*]

2 Systems of particles

A mathematical model consisting of more than one particle is referred to as a *system* of particles. Every such system is enclosed within a notional boundary. Forces due to items inside the system boundary are called *internal forces* of the system, whereas forces whose cause lies beyond the system boundary are called *external forces*.

For example, Figure 1(a) shows a system of two particles. In order to indicate that both particles belong to the system, the boundary may be drawn around them as shown by the broken lines in Figure 1(b). The forces acting on the particles due to the perfect spring of stiffness k_2 are internal forces, because this spring joins the particles and is therefore inside the system, but the force acting on the particle of mass m_1 due to the perfect spring of stiffness k_1 is an external force, since this spring lies outside the system. The gravitational forces on the particles are due to the attraction of the Earth, and since the Earth is not inside the system boundary, these gravitational forces are external to the system.

It is not always necessary to draw in the system boundary, when the composition of the system is clearly understood, and it may not be essential to distinguish between the internal and external forces as we did in the example above. However, this terminology is helpful in some situations, and the system boundary is a useful modelling concept.



(a) (b)

Figure 1

2.1 Two-particle spring systems

When two particles are joined by a taut string and each is constrained to move parallel to the local part of the string, as in Section 1, then the speeds of the particles must be the same at any given time. Consequently, the magnitudes of their accelerations are also the same. This is not so for a two-particle system in which the particles are joined by a spring rather than a string. However, the same procedure as was applied earlier to string systems will also give some information about the motion of a spring system.

We shall look first at the system of two particles shown in Figure 2, where there are no net external forces, before going on to consider a case which also involves external forces. In the next subsection the technique used to investigate these systems will be generalized to other two-particle systems.

The system in Figure 2 consists of two particles joined by a perfect spring but otherwise free to move on a frictionless horizontal track. The masses of the particles are m_1 and m_2 respectively, while the perfect spring has natural length l_0 and stiffness k. We assume for the analysis that the spring is extended but, as you saw in *Unit 7*, the resulting equations of motion will be valid also when the spring is compressed. The x-axis is chosen to point from left to right, and the positions of the particles relative to the fixed origin O are denoted by x_1 and x_2 respectively.

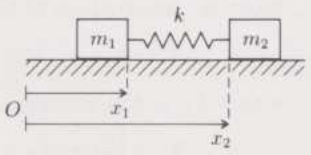


Figure 2

The particles are constrained to move horizontally (and indeed in one dimension), so the gravitational forces are balanced by the normal reactions and only the horizontal forces on the particles need be considered. The total force acting on each particle is the spring force, as shown in Figure 3.

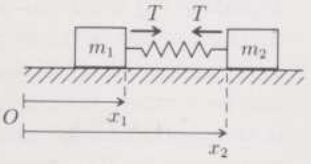


Figure 3

We now apply Newton's second law to each particle in order to obtain its equation of motion. For the first particle this gives

$$m_1\ddot{x}_1 = T, \tag{1}$$

while for the second particle we obtain

$$m_2\ddot{x}_2 = -T. \tag{2}$$

The right-hand sides of these differential equations have the same magnitude but opposite signs, because the corresponding spring forces are of equal magnitude but act in opposite directions. Thus the variable T may be eliminated by adding these equations, which produces

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0. \tag{3}$$

This equation provides information about a point of the system distinct from each of the two individual particles but defined in terms of them, namely, their centre of mass. Suppose that the total mass $m_1 + m_2$ of the system is denoted by M. Then the **centre of mass** of the system is defined to be the point whose position X is given by

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1}{M}x_1 + \frac{m_2}{M}x_2. \tag{4}$$

Note that m_i/M $(i = 1, 2)$ is the proportion of the total mass concentrated at the ith particle, so that X is a weighted average of the positions of the two particles. In fact, it lies always between the two particles. This can be seen more clearly by replacing one of x_1, x_2 using the relation $l = x_2 - x_1$, where l is the length of the spring. Then Equation (4) can be rewritten in the alternative forms

$$X = x_1 + \frac{m_2}{M}l = x_2 - \frac{m_1}{M}l, \tag{5}$$

as illustrated in Figure 4(a) below. If the particles are of equal mass then X is midway between the particles, as shown in Figure 4(b). If one particle is heavier than the other then the centre of mass is nearer to the heavier particle, as illustrated in Figure 4(c) for the case in which Particle 1 is twice the mass of Particle 2.

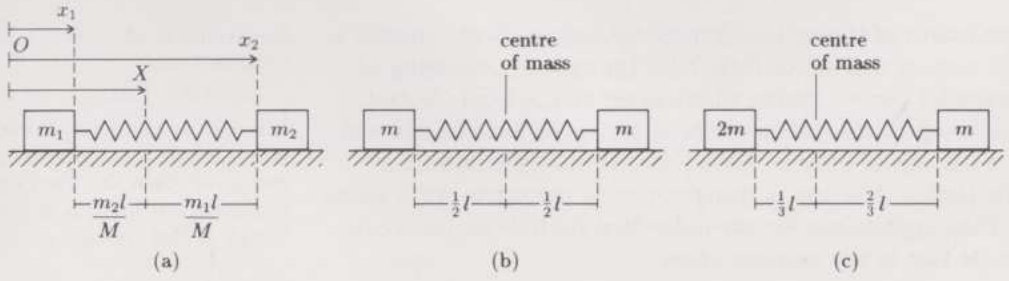


Figure 4

The centre of mass is important because its motion is simple to calculate. Equation (3) can be re-expressed in terms of X as $M\ddot{X} = 0$, or just as

$$\ddot{X} = 0.$$

This indicates that the centre of mass has zero acceleration and hence is in uniform motion. Integration of this equation gives

$$\dot{X} = V_0,$$

where V_0 is the constant velocity of the centre of mass. A second integration produces

$$X = V_0 t + X_0, \tag{6}$$

where X_0 is the initial position of the centre of mass. Note that the centre of mass motion has been found without using any knowledge of how the magnitude T of the internal forces depends upon the particle positions. We used only the fact that the two forces concerned were of equal magnitude but acted in opposite directions.

Equation (6) describes how the centre of mass moves, but it tells us nothing about the motions of the particles at either end of the spring, that is, about the position functions $x_1(t)$ and $x_2(t)$. It is clear from Equations (5) that expressions for these two functions can be written down once we have found the functions $X(t)$ and $l(t)$. The first of these is the centre of mass motion given by Equation (6), so we seek next to derive an expression for $l(t)$, which describes how the length of the spring varies with time.

This length is given by $l = x_2 - x_1$, and taking the derivative of this equation twice with respect to time produces

$$\ddot{l} = \ddot{x}_2 - \ddot{x}_1.$$

By applying Equations (1) and (2) we then obtain

$$\begin{aligned} \ddot{l} &= -\frac{T}{m_2} - \frac{T}{m_1} \\ &= -\frac{T(m_1 + m_2)}{m_1 m_2}. \end{aligned}$$

This may be rewritten as

$$\mu \ddot{l} = -T, \tag{7}$$

where the **reduced mass** μ is defined by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{8}$$

There is no connection here with the previous use of the symbol μ to denote a coefficient of friction.

Here μ is referred to as a *mass* because it has the dimensions of mass, and as *reduced* since it is necessarily smaller than either of m_1 and m_2.

You should find it straightforward to show that

$$\mu < m_i \quad (i = 1, 2).$$

From Equation (7) we proceed by expressing T in terms of the spring length l. The spring is assumed to be perfect so that Hooke's law applies, giving

$$T = k(l - l_0),$$

Unit 7 Subsection 2.1

where k is the stiffness of the spring and l_0 is its natural length. Then Equation (7) becomes

$$\mu \ddot{l} + kl = kl_0. \tag{9}$$

This equation governs how the length of the spring changes with time. Notice that it is very similar to the equation of motion derived in *Unit 7* for the system consisting of one particle attached to a horizontal perfect spring whose other end is fixed. In fact, the length of the spring in our two-particle system varies as if it were fixed at one end and attached at the other to a particle of mass μ. Given that this simple harmonic motion equation was solved in *Unit 7*, it is now a straightforward matter to write down the general solution for $l(t)$. Then expressions for the individual particle motions can be deduced. You are asked to do this in the exercise below.

Equation (9) of *Unit 7* Section 2 was

$$m\ddot{x} + kx = kl_0,$$

where m is the mass of the particle and x is its position measured from the fixed end of the spring (that is, x is the length of the spring).

Exercise 1

(i) By reference to *Unit 7* Subsection 2.3, or otherwise, write down the general solution of Equation (9).

(ii) Use the result of part (i) together with Equations (5) and (6) to deduce expressions for the position functions $x_1(t)$ and $x_2(t)$ of the two particles.

[*Solution on page 48*]

In Exercise 1 you showed that the positions of the individual particles are given in terms of the position X of the centre of mass and the length l of the spring by

$$x_1 = X - \frac{m_2}{M}l \quad \text{and} \quad x_2 = X + \frac{m_1}{M}l. \tag{10}$$

Here we have $X = V_0t + X_0$ from Equation (6) and

$$l = l_0 + A\cos(\omega t + \phi), \tag{11}$$

where $\omega = \sqrt{k/\mu}$. All solutions of this form are obtainable with A non-negative and ϕ between $-\pi$ and π. Then A and ϕ are respectively the amplitude and phase of the spring length oscillation.

So far we have derived the general solutions for X, l, x_1 and x_2. In order to obtain a particular solution we need to use given initial conditions to find the values of the constants V_0, X_0 (for X) and A, ϕ (for l). There are various ways in which the initial conditions might be specified, but in each case the simplest approach is to deduce the corresponding conditions for X and l if this is not the information first provided. The example below illustrates the procedure required.

Example 1

Consider the system shown in Figure 4(b), where $m_1 = m_2 = m$. Suppose that the system is initially released from rest at time $t = 0$ when the length of the spring is $\frac{3}{2}l_0$. Choosing the origin O to be at the initial position of Particle 1 (the left-hand particle) and the x-axis to be directed from O towards Particle 2, find the particular solutions of the equations of motion for X, l, x_1 and x_2.

Solution

Note first that, for the system depicted in Figure 4(b), the total mass is $M = 2m$, the reduced mass is $\mu = \frac{1}{2}m$ (from Equation (8)) and the centre of mass position is given by $X = \frac{1}{2}(x_1 + x_2)$ (from Equation (4)). Also the length of the spring is given as before by $l = x_2 - x_1$. In terms of x_1 and x_2 the initial conditions are

$$x_1(0) = 0,\ \dot{x}_1(0) = 0 \quad \text{and} \quad x_2(0) = \tfrac{3}{2}l_0,\ \dot{x}_2(0) = 0.$$

Thus the initial conditions for l and X are

$$l(0) = \tfrac{3}{2}l_0,\ \dot{l}(0) = 0 \quad \text{and} \quad X(0) = \tfrac{3}{4}l_0,\ \dot{X}(0) = 0.$$

Now from Equation (6) we have $X_0 = X(0)$ and $V_0 = \dot{X}(0)$, so that $X = V_0t + X_0$ becomes

$$X = \tfrac{3}{4}l_0.$$

From Equation (11) we obtain

$$l(0) = l_0 + A\cos\phi, \qquad \dot{l}(0) = -\omega A\sin\phi,$$

and from the initial conditions for l we can therefore deduce that $\phi = 0$ and $A = \frac{1}{2}l_0$. The particular solution for l is then

$$l = l_0 + \tfrac{1}{2}l_0\cos\omega t,$$

where $\omega = \sqrt{k/\mu} = \sqrt{2k/m}$. From Equations (10) the corresponding particular solutions for the individual particles are

$$\begin{aligned} x_1 &= \tfrac{3}{4}l_0 - \tfrac{1}{2}(l_0 + \tfrac{1}{2}l_0\cos\omega t) \\ &= \tfrac{1}{4}l_0 - \tfrac{1}{4}l_0\cos\omega t \end{aligned}$$

and

$$\begin{aligned} x_2 &= \tfrac{3}{4}l_0 + \tfrac{1}{2}(l_0 + \tfrac{1}{2}l_0\cos\omega t) \\ &= \tfrac{5}{4}l_0 + \tfrac{1}{4}l_0\cos\omega t. \end{aligned}$$

The particles oscillate with the same angular frequency $\omega = \sqrt{2k/m}$ and amplitude $\frac{1}{4}l_0$ about the respective positions $\frac{1}{4}l_0$ and $\frac{5}{4}l_0$, while the centre of mass remains at the fixed position $\frac{3}{4}l_0$. There is a phase difference of π between the particle motions, as is seen on rewriting the expression for x_1 as

$$x_1 = \tfrac{1}{4}l_0 + \tfrac{1}{4}l_0\cos(\omega t + \pi).$$

This means that x_1 attains its maximum value at the same moment that x_2 attains its minimum (when the spring is at its shortest) and vice versa (when the spring is at its longest). □

Exercise 2

Consider once more the system of Figure 4(b) on page 11. Suppose that initially (at time $t = 0$) the length of the spring is its natural length, Particle 1 is at rest (at the origin O) and Particle 2 is moving away from O with speed v_0. As before, take the x-axis to be directed from O towards Particle 2.

(i) Find the subsequent motions of the system's centre of mass and of each of the particles.

(ii) In order for the expressions derived in part (i) to be valid within the model for all $t > 0$, it is necessary that the condition $v_0 < l_0\sqrt{2k/m}$ should be satisfied. Explain why this condition is required.

[*Solution on page 49*]

Each particular motion of our two-particle spring system consists of a uniform motion for the centre of mass X together with a simple harmonic motion for the spring length l. Thus the system as a whole (as represented by its centre of mass) has constant velocity, while the two particles within it are oscillating. Each individual particle motion is a linear combination of the motions for X and l, and is a simple harmonic oscillation about a point whose position is fixed relative to the centre of mass. Most points on the spring move in a similar way to the particles at either end. Provided that the amplitude A is non-zero, the centre of mass is the only point which does not oscillate at all.

As you will see, several aspects of this extended example apply in other situations. This applies in particular to the process of combining the equations of motion for the individual particles in order to obtain the equation of motion for the centre of mass, which was possible because the two internal forces were equal in magnitude but opposite in direction.

In the following exercises you are invited to investigate how the previous analysis is affected by the presence of external forces.

Exercise 3

The two-particle spring system in Figure 5 is the same as that considered previously, except that now an external force with x-component F acts upon Particle 1. By suitably modifying the equations of motion derived earlier for the particles (Equations (1) and (2)), show that the equation of motion for the centre of mass X in this new system is

$$M\ddot{X} = F,$$

where $M = m_1 + m_2$ is the total mass of the system.

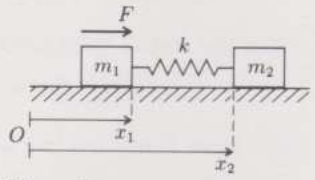


Figure 5

Exercise 4

Figure 6 shows a system consisting of two particles joined by a vertical spring and falling vertically. Both the internal forces acting on the particles (due to the spring) and the external forces (due to gravity) are indicated on the diagram. The x-axis is taken to point downwards. Find the equation of motion for the centre of mass of this system.

[*Solutions on page 49*]

For each of the systems considered in the two exercises above, the equation of motion for the centre of mass X can be written as

$$(\text{total mass})\ddot{X} = \text{total external force},$$

where the total mass is the sum of the individual particle masses, and the total external force is the algebraic sum of the external forces acting upon the particles. The same form of equation is satisfied by the original system where only internal forces were present, since this equation reduces to $\ddot{X} = 0$ in the absence of external forces.

In other words, the motion of the centre of mass is identical to that of a particle with mass equal to the total mass of the system, which is acted upon simultaneously by all the external forces present. This is a particular version (for a two-particle spring system) of a more general result, as you will see later in the current section.

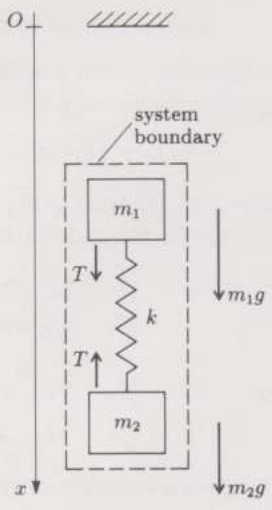


Figure 6

2.2 Two-particle systems and Newton's third law

In the two-particle systems considered so far, the internal forces have been transmitted through a specific device, like a string or spring. Another article capable of force transmission is a dashpot. However, many forces act without the mediation of any physical connection. Two such examples are magnetic force, as you will see in the television programme, and gravitational force.

If two particles exert forces upon each other then their motion can be analysed in the same way as for the systems already considered, whether or not the forces are transmitted by a connecting device. To some extent such systems behave similarly to the two-particle spring systems investigated in the last subsection. All of the internal forces acting within a system of particles may be regarded as forces between the individual particles, and to emphasize this we shall from now on refer to these forces as **inter-particle forces**.

Our analysis of the two-particle spring system led to a relatively simple equation of motion for the system's centre of mass. The simplicity of this equation depends on the fact that the forces exerted mutually by the particles upon each other (via the medium of the spring) are equal in magnitude but opposite in direction. For a general two-particle system we assume that each pair of forces between particles obeys the same rule. This is an application to particles of **Newton's third law**, which was stated by Newton in the *Principia* as follows:

> To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

This law was stated, together with Newton's first and second laws, in the Introduction to *Unit 4*. It has not been mentioned further prior to this unit, since we were previously dealing only with one-particle systems.

Our current aim is to obtain the equation of motion for the centre of mass of any two-particle system, given that the inter-particle forces satisfy Newton's third law.

In general, the motion of a two-particle system can take place in two or three dimensions, rather than in the single dimension of the model in Subsection 2.1. This means that we shall need to call upon the vector notation of *Unit 15*, noting that in three dimensions position, velocity, acceleration and force are vector quantities. Thus if $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the axes of a static right-handed Cartesian coordinate system, and x, y and z are the corresponding Cartesian coordinates of a particle (each of which may vary with time as the particle moves), then the particle has position vector

Any result derived for a system of particles in three dimensions holds also for a one- or two-dimensional system.

These definitions were introduced in *Unit 15* Section 1.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

velocity vector

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

and acceleration vector

$$\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}.$$

Since we are now considering the motion of a *pair* of particles, we must add subscripts to $\mathbf{R}$, x, y, z and their derivatives to indicate which particle is being described. For example, the position vector for Particle 1 is written as

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k},$$

whereas that for Particle 2 is similar but with the subscript 1 replaced by 2 throughout.

In order to proceed we require the following to be expressed in terms of vectors:

(i) a definition of the centre of mass of any two-particle system;

(ii) a statement of Newton's third law.

In each case, the vector expression will be the natural generalization to three dimensions of what you have already seen in one dimension.

Figure 7 shows a general two-particle system in three dimensions. The particles have masses m_1 and m_2 respectively, and position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. By analogy with the one-dimensional case, the **centre of mass** of the system is defined to be the point with position vector

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \tag{12}$$

where $M = m_1 + m_2$ is the total mass of the system. If the position of the centre of mass has Cartesian coordinates X, Y and Z then this equation can be written as

$$X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \frac{m_1x_1 + m_2x_2}{M}\mathbf{i} + \frac{m_1y_1 + m_2y_2}{M}\mathbf{j} + \frac{m_1z_1 + m_2z_2}{M}\mathbf{k},$$

which is equivalent to the three component equations

$$X = \frac{m_1x_1 + m_2x_2}{M}, \quad Y = \frac{m_1y_1 + m_2y_2}{M}, \quad Z = \frac{m_1z_1 + m_2z_2}{M}.$$

If both particles happen to lie on the x-axis, that is, if $y_1 = y_2 = z_1 = z_2 = 0$, then we have $Y = Z = 0$, showing that the centre of mass also lies on the x-axis. The expression for X is just that given by Equation (4), which confirms that our three-dimensional definition is indeed the natural extension of the previous one-dimensional definition. As in the one-dimensional case, it can be shown that the centre of mass as defined by Equation (12) always lies on the line segment which joins the two particles. The next exercise asks you to locate the centre of mass of a particular system.

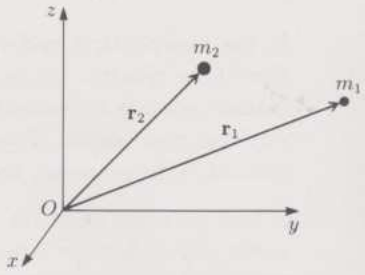


Figure 7

Exercise 5

A model of a dumb-bell consists of two particles joined together by a rigid rod of negligible mass. The first particle has mass 3 kg and is situated at the origin. The second particle has mass 1 kg and is situated at the point with position vector $0.3\mathbf{i} + 0.4\mathbf{j}$.

(i) Draw a diagram showing the location of the particles.

(ii) Find the position vector of the centre of mass.

(iii) On your diagram from part (i), show the position of the centre of mass.

[*Solution on page 49*]

Having defined the centre of mass for a two-particle system in vector terms, we turn to the task of writing down a mathematical formulation of Newton's third law. Figure 8 shows a pair of particles with no external forces acting on them. The force vectors shown represent the inter-particle forces which each particle exerts on the other. If it assists you to consider a specific example, you might like to think of Figure 8 as representing a pair of pucks which repel each other magnetically while moving with negligible friction on a horizontal surface (this is a situation which you will see demonstrated in the television programme). Having said that, the forces could in the general case be either repulsive or attractive at any given time.

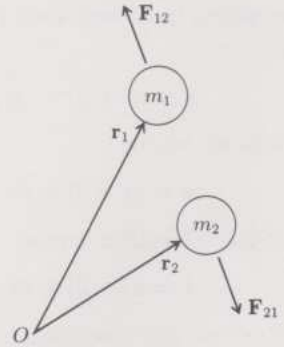


Figure 8

Figure 8 illustrates a vector notation with double subscripts for describing inter-particle forces. In this notation the first subscript to the vector symbol $\mathbf{F}$ represents the particle which experiences the force, whereas the second subscript indicates the particle which exerts the force. Thus the force on Particle 1 due to the effect of Particle 2 is denoted by the vector $\mathbf{F}_{12}$ and, similarly, the force on Particle 2 due to the effect of Particle 1 is denoted by $\mathbf{F}_{21}$. This notation may appear cumbersome for a system of two particles, but it can be applied with equal ease to larger systems, as you will see shortly.

We are now in a position to express Newton's third law in vector form. This law states that the inter-particle forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are

(i) equal in magnitude,

(ii) opposite in direction.

Taken together, these two conditions are equivalent to requiring that each of the two force vectors should be minus the other.

Since the two forces are equal in magnitude and opposite in direction, they are sometimes referred to as being *equal and opposite* forces. This is, strictly speaking, a contradictory phrase, but it has the merit of brevity.

Newton's third law in vector form

Suppose that inter-particle forces act between two particles labelled 1 and 2. The force on Particle 1 due to the effect of Particle 2 is denoted by the vector $\mathbf{F}_{12}$, and the force on Particle 2 due to the effect of Particle 1 is represented by the vector $\mathbf{F}_{21}$. Then *Newton's third law* states that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{13}$$

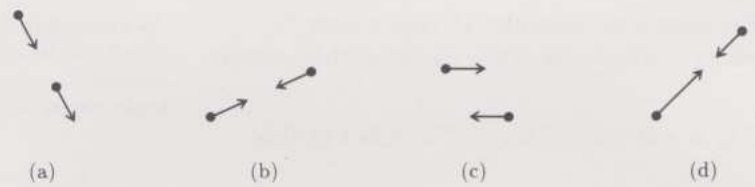
Note that this vector form of Newton's third law applies to the two-particle spring system considered in Subsection 2.1. Corresponding to the inter-particle forces shown in Figure 3, which represents the situation in which the spring is extended, we have

$$\mathbf{F}_{12} = T\mathbf{i} \quad \text{and} \quad \mathbf{F}_{21} = -T\mathbf{i},$$

so that Equation (13) is satisfied.

Exercise 6

(i) Figure 9 shows four different two-particle systems. The inter-particle force vectors are represented by arrows where, as usual, the length of each arrow is proportional to the magnitude of the force represented, and the direction of the arrow is that of the force. All the particles and forces lie in the plane of the page. Which of the pairs of inter-particle forces are consistent with Newton's third law?



(a) (b) (c) (d)

Figure 9

(ii) Do the string forces which act on the particles in the systems of Section 1 satisfy Newton's third law?

[*Solution on page 49*]

Having defined the centre of mass for a two-particle system in three dimensions and stated Newton's third law in vector form, we proceed to analyse the motion of a two-particle system. Suppose first that no external forces act on the particles. Then the treatment required is an extension to three dimensions of the one-dimensional case developed through Equations (1) to (6).

The system is as depicted in Figure 8, and we start by applying Newton's second law in its vector form to each of the particles. For Particle 1 this gives

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$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{12},$$

while for Particle 2 we have

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{21}.$$

Now the inter-particle forces are assumed to satisfy Newton's third law (Equation (13)), which is equivalent to $\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0}$. Hence on adding the last two equations, we obtain

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}. \tag{14}$$

This can be rewritten in terms of the total mass $M = m_1 + m_2$ and the position vector $\mathbf{R}$ of the centre of mass. From Equation (12) we have

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M},$$

so that Equation (14) can be expressed as $M\ddot{\mathbf{R}} = \mathbf{0}$, or just as

$$\ddot{\mathbf{R}} = \mathbf{0}.$$

This indicates that the centre of mass has zero acceleration and is therefore in uniform motion. Integration of this equation gives

$$\dot{\mathbf{R}} = \mathbf{V}_0,$$

where $\mathbf{V}_0$ is the constant velocity of the centre of mass. A second integration produces

$$\mathbf{R} = \mathbf{V}_0 t + \mathbf{R}_0, \tag{15}$$

where $\mathbf{R}_0$ is the initial position of the centre of mass. As in the one-dimensional case considered earlier, the centre of mass motion has been derived without any detailed knowledge of how the inter-particle forces behave. We used only the fact that this pair of forces satisfies Newton's third law.

In order to investigate the motions of the individual particles, information about the specific form of the inter-particle forces must be provided, as in the case of the two-particle spring system. However, in some situations predictions of a limited kind can be made on the basis of partial knowledge of how the system moves. Such predictions usually depend on finding the motion of the centre of mass, which is a simple task compared with that of locating the particles themselves. The next example and exercise demonstrate the sort of calculation which is involved.

Example 2

Two particles move in the (x, y)-plane. They repel each other by inter-particle forces which obey Newton's third law, and they experience no external forces.

Particle 1 has mass 3 kg and Particle 2 has mass 2 kg. Initially (at time $t = 0$), Particle 1 has position $\mathbf{r}_1(0) = \mathbf{0}$ and velocity $\dot{\mathbf{r}}_1(0) = -4\mathbf{i}$, while Particle 2 has position $\mathbf{r}_2(0) = 3\mathbf{j}$ and velocity $\dot{\mathbf{r}}_2(0) = 6\mathbf{i}$.

As previously, all physical quantities in this unit are to be taken as measured in SI units unless stated otherwise.

At time $t = 1$, Particle 1 is located at $\mathbf{r}_1(1) = -3\mathbf{i} + 2\mathbf{j}$. Where is Particle 2 at this time?

Solution

From Equation (12), the position and velocity of the centre of mass at time $t = 0$ are

$$\mathbf{R}(0) = \frac{m_1\mathbf{r}_1(0) + m_2\mathbf{r}_2(0)}{m_1 + m_2} = \frac{3 \times \mathbf{0} + 2 \times 3\mathbf{j}}{3 + 2} = 1.2\mathbf{j}$$

and

$$\dot{\mathbf{R}}(0) = \frac{m_1\dot{\mathbf{r}}_1(0) + m_2\dot{\mathbf{r}}_2(0)}{m_1 + m_2} = \frac{3 \times (-4\mathbf{i}) + 2 \times 6\mathbf{i}}{3 + 2} = \mathbf{0}.$$

Hence in Equation (15) we have $\mathbf{R}_0 = 1.2\mathbf{j}$ and $\mathbf{V}_0 = \mathbf{0}$, giving $\mathbf{R}(t) = 1.2\mathbf{j}$. In particular, we have $\mathbf{R}(1) = 1.2\mathbf{j}$. However, from the position given for Particle 1 at time $t = 1$, we also know that

$$\mathbf{R}(1) = \frac{m_1\mathbf{r}_1(1) + m_2\mathbf{r}_2(1)}{m_1 + m_2} = \frac{3(-3\mathbf{i} + 2\mathbf{j}) + 2\mathbf{r}_2(1)}{3 + 2} = \frac{-9\mathbf{i} + 6\mathbf{j} + 2\mathbf{r}_2(1)}{5}.$$

On equating the two expressions for $\mathbf{R}(1)$, we obtain

$$\mathbf{r}_2(1) = \tfrac{1}{2}(9\mathbf{i} - 6\mathbf{j} + 5 \times 1.2\mathbf{j}) = 4.5\mathbf{i}.$$

Thus at time $t = 1$, Particle 2 is located 4.5 m along the positive x-axis. □

Exercise 7

Two particles of equal mass move in the (x, y)-plane. They exert upon each other inter-particle forces which obey Newton's third law, and they experience no external forces. Initially (at time $t = 0$), Particle 1 has position $\mathbf{r}_1(0) = 2\mathbf{j}$ and velocity $\dot{\mathbf{r}}_1(0) = -\mathbf{j}$, while Particle 2 has position $\mathbf{r}_2(0) = -2\mathbf{j}$ and velocity $\dot{\mathbf{r}}_2(0) = 4\mathbf{i}$.

At time $t = 2$, Particle 1 is located at $\mathbf{r}_1(2) = \mathbf{i}$ and has velocity $\dot{\mathbf{r}}_1(2) = \mathbf{i} - \mathbf{j}$. Find the position and velocity of Particle 2 at this time.

[*Solution on page 50*]

We have shown that, in the absence of external forces, the centre of mass $\mathbf{R}$ of a two-particle system satisfying Newton's third law has the equation of motion $\ddot{\mathbf{R}} = \mathbf{0}$. This is the three-dimensional counterpart of the equation $\ddot{X} = 0$ obtained for the one-dimensional system in Subsection 2.1. We demonstrated there that, when external forces are taken into account, the equation of motion of the centre of mass X becomes

$$(\text{total mass})\ddot{X} = \text{total external force},$$

where the total external force is the algebraic sum of the external forces acting upon the particles. You might then anticipate that we could again reach the corresponding three-dimensional equation on replacing $\ddot{X}$ by $\ddot{\mathbf{R}}$. This is in fact so, provided that 'total external force' is now interpreted as the vector sum of the external forces acting on the individual particles. You are asked to verify this result in the exercise below.

Exercise 8

Figure 10 shows a two-particle system, with the system boundary drawn in to distinguish more clearly between the internal (inter-particle) and external forces. The two particles have masses m_1, m_2 and respective position vectors $\mathbf{r}_1$, $\mathbf{r}_2$. Particle 1 is acted upon by an inter-particle force $\mathbf{F}_{12}$ and an external force $\mathbf{F}_1^{\text{ext}}$, while Particle 2 experiences an inter-particle force $\mathbf{F}_{21}$ and an external force $\mathbf{F}_2^{\text{ext}}$.

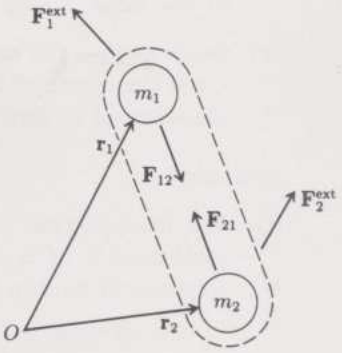


Figure 10

(i) Apply Newton's second law to each of the particles in turn.

(ii) Assuming that the inter-particle forces satisfy Newton's third law, combine your results from part (i) to show that the centre of mass $\mathbf{R}$ of the system has the equation of motion

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}},$$

where $M = m_1 + m_2$ is the total mass and $\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}$ is the total external force.

[*Solution on page 50*]

The equation of motion which you derived in part (ii) of this exercise has the same form as that for a single particle of mass M and position vector $\mathbf{R}$, acted upon by a force $\mathbf{F}^{\text{ext}}$. This means that the centre of mass moves as if all the mass of the system were concentrated in a particle at that point which is acted upon simultaneously by all of the external forces. In the absence of external forces we have $\mathbf{F}^{\text{ext}} = \mathbf{0}$, and the equation of motion of the centre of mass reduces to $\ddot{\mathbf{R}} = \mathbf{0}$, as derived previously.

You will see shortly that similar results hold for a system consisting of more than two particles. The results of this subsection are summarized below.

Motion of the centre of mass of a two-particle system

For a system of two particles with masses m_1, m_2 and respective position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, the *centre of mass* has position vector

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \tag{12}$$

where $M = m_1 + m_2$ is the *total mass* of the system. If the particles are acted upon by external forces $\mathbf{F}_1^{\text{ext}}$ and $\mathbf{F}_2^{\text{ext}}$ respectively, and if the inter-particle forces satisfy Newton's third law, then the equation of motion of the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}, \tag{16}$$

where $\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}$ is the *total external force* acting on the system. In the absence of external forces, Equation (16) reduces to $\ddot{\mathbf{R}} = \mathbf{0}$, indicating a uniform motion for the centre of mass.

As in the case with no external forces, Equation (16) has been derived without any knowledge of the inter-particle force behaviour beyond the assumption that Newton's third law is satisfied. Further information about the inter-particle forces is needed before the individual particle motions may be sought. Nevertheless, if Equation (16) can be solved for $\mathbf{R}(t)$ and a complete set of initial conditions is provided, then it is possible to predict one particle's position or velocity at a given time from knowledge of the corresponding data for the other particle. You saw in Example 2 and Exercise 7 how this could be achieved where there are no external forces. The following example and exercise indicate that the process is similar when external forces are present.

Example 3

Figure 11 shows a two-particle spring system falling freely under gravity. The particles have equal mass m and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$. The x-axis is directed vertically downwards, so that each particle experiences an external force $mg\mathbf{i}$ due to gravity. The inter-particle forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ satisfy Newton's third law.

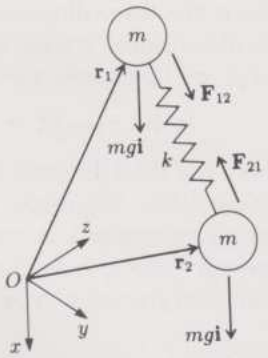


Figure 11

(i) Find the equation of motion for the centre of mass $\mathbf{R}$ of this system.

(ii) Find the general solution of the differential equation derived in part (i).

(iii) Initially (at time $t = 0$), Particle 1 is at the origin and has velocity $3\mathbf{j}$, while Particle 2 has position $4\mathbf{i}$ and velocity $\mathbf{j}$. Find the corresponding motion of the centre of mass.

(iv) At time $t = 2$, Particle 1 has position $24\mathbf{i} + 3\mathbf{j}$. What is the position of Particle 2 at this time? (Take $g = 9.8\,\mathrm{m\,s^{-2}}$.)

(v) Suppose that the spring is perfect, with stiffness $k = 20\,\mathrm{N\,m^{-1}}$, and that at time $t = 0$ the length of the spring is its natural length. Find the inter-particle force experienced at time $t = 2$ by Particle 2.

Solution

(i) The inter-particle forces satisfy Newton's third law, so Equation (16) applies. The total mass is $M = 2m$, and the total external force is $\mathbf{F}^{\mathrm{ext}} = 2mg\mathbf{i}$. Hence the equation of motion for the centre of mass is

$$\ddot{\mathbf{R}} = g\mathbf{i}.$$

(Note that this is essentially the same as the equation derived for the motion of a shot in the first model of *Unit 15* Section 4, showing that the centre of mass here moves in the same way as the shot, that is, along a parabolic path. Note also that for the one-dimensional version of this system, where $\mathbf{R} = X\mathbf{i}$, the equation of motion is just $\ddot{X} = g$, as you found in Exercise 4.)

(ii) Since $g\mathbf{i}$ is a constant vector, integration of the equation in part (i) gives

$$\dot{\mathbf{R}} = gt\mathbf{i} + \mathbf{V}_0,$$

where the constant vector $\mathbf{V}_0$ is the velocity of the centre of mass at time $t = 0$. A further integration produces

$$\mathbf{R} = \tfrac{1}{2}gt^2\mathbf{i} + \mathbf{V}_0 t + \mathbf{R}_0,$$

where the constant vector $\mathbf{R}_0$ is the position of the centre of mass at time $t = 0$. This is the required general solution. (It is possible also to solve the differential equation by first expressing it in component form, which was the approach adopted in *Unit 15* Section 4.)

(iii) The initial conditions for the two particles are

$$\mathbf{r}_1(0) = \mathbf{0}, \quad \dot{\mathbf{r}}_1(0) = 3\mathbf{j}$$

and $$\mathbf{r}_2(0) = 4\mathbf{i}, \quad \dot{\mathbf{r}}_2(0) = \mathbf{j}.$$

Since $M = 2m$, Equation (12) becomes

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2),$$

from which we also obtain

$$\dot{\mathbf{R}} = \tfrac{1}{2}(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2).$$

The initial conditions for the centre of mass are therefore

$$\mathbf{R}(0) = 2\mathbf{i}, \quad \dot{\mathbf{R}}(0) = 2\mathbf{j}.$$

Hence in the general solution of part (ii) we have $\mathbf{R}_0 = 2\mathbf{i}$ and $\mathbf{V}_0 = 2\mathbf{j}$, giving the particular solution

$$\mathbf{R} = (\tfrac{1}{2}gt^2 + 2)\mathbf{i} + 2t\mathbf{j}.$$

(iv) At time $t = 2$, the positions of the centre of mass and of Particle 1 are respectively

$$\mathbf{R}(2) = 2(g+1)\mathbf{i} + 4\mathbf{j} \quad \text{and} \quad \mathbf{r}_1(2) = 24\mathbf{i} + 3\mathbf{j},$$

so the position of Particle 2 at this time is

$$\begin{aligned}\mathbf{r}_2(2) &= 2\mathbf{R}(2) - \mathbf{r}_1(2) \\ &= 4(g-5)\mathbf{i} + 5\mathbf{j} = 19.2\mathbf{i} + 5\mathbf{j}.\end{aligned}$$

(v) We seek the inter-particle force $\mathbf{F}_{21}(2)$. The natural length of the spring is

$$|\mathbf{r}_2(0) - \mathbf{r}_1(0)| = |4\mathbf{i} - \mathbf{0}| = 4\,\text{m},$$

and at time $t = 2$ its length is

$$\begin{aligned}|\mathbf{r}_2(2) - \mathbf{r}_1(2)| &= |-4.8\mathbf{i} + 2\mathbf{j}| \\ &= \sqrt{(-4.8)^2 + 2^2} = 5.2\,\text{m}.\end{aligned}$$

The spring is then extended by 1.2 m, so by Hooke's law the magnitude of $\mathbf{F}_{21}(2)$ is

$$20 \times 1.2 = 24\,\text{N}.$$

Since the spring is extended, this force is directed from $\mathbf{r}_2(2)$ towards $\mathbf{r}_1(2)$, that is, in the direction of the unit vector

$$\frac{\mathbf{r}_1(2) - \mathbf{r}_2(2)}{|\mathbf{r}_1(2) - \mathbf{r}_2(2)|} = \frac{4.8\mathbf{i} - 2\mathbf{j}}{5.2} = \frac{12\mathbf{i} - 5\mathbf{j}}{13}.$$

Hence the required inter-particle force is

$$\mathbf{F}_{21}(2) = \tfrac{24}{13}(12\mathbf{i} - 5\mathbf{j}). \quad \square$$

Exercise 9

Consider once more the system of Exercise 7, but suppose now that the x-axis is directed vertically downwards and that each particle experiences an external force $mg\mathbf{i}$ due to gravity. Solve the resulting modified version of Exercise 7, where the remaining data of the problem are unchanged. (Take $g = 9.8\,\text{m}\,\text{s}^{-2}$.)

[*Solution on page 50*]

2.3 Many-particle systems

We now extend the results of Subsection 2.2 to systems which consist of more than two particles. Indeed, the number of particles involved may be large in a practical model. For example, a solid body is made up of many molecules, each of which might be modelled by a particle. Alternatively, a particle could represent each atom in a model of a molecule, or we could consider a model of the atom featuring sub-atomic particles.

An object of given extent and mass is in a sense better represented by an appropriate model involving many particles than it is by one which has fewer particles, since physical shape, size and consistency can be described more and more closely as the number of representative particles is increased. In this sense the single-particle model of *Units 4*, *7*, *8* and *15* is the crudest available, and we are now seeking to refine it. Despite this, the single-particle model continues to be of use. As you are about to see, the centre of mass of a many-particle system has the same equation of motion as the appropriate single-particle model.

With a greater number of particles to represent a given object, there will be on average a decrease in the mass of each particle, in order that the overall mass should remain the same. Another way of putting this is that with more particles present, each particle tends to represent a smaller proportion of the whole. The process of increasing the number of representative particles while reducing the mass of each can be carried to a limit in which we reach an alternative model of continuous matter distribution. You will see this type of model considered in *Units 27* and *29*, but it is worth noting in advance that the results of the current subsection apply in the limit to the continuous-matter model.

Consider then a system which consists of n particles, where n is any positive integer. The ith particle has mass m_i and position vector $\mathbf{r}_i$ $(i = 1, 2, \ldots, n)$. We start by extending the previous notation for inter-particle forces. Suppose that i, j are distinct

numbers in the set $\{1, 2, \ldots, n\}$. Then the inter-particle force on Particle i due to Particle j is denoted by $\mathbf{F}_{ij}$. By the same token, the inter-particle force on Particle j due to Particle i is denoted by $\mathbf{F}_{ji}$. This notation is illustrated in Figure 12.

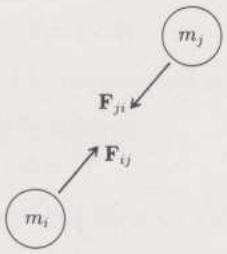


Figure 12

In an n-particle system each particle has $n-1$ inter-particle forces acting upon it, one for each of the other particles in the system. This makes a total of $n(n-1)$ inter-particle forces within the system. We continue to assume that Newton's third law applies, so that the inter-particle forces between Particles i and j satisfy the equation

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \qquad (i, j = 1, 2, \ldots, n;\ i \neq j). \tag{17}$$

This is a restatement of Equation (13), but applies now to any pair of particles within the n-particle system.

We also need to extend from Equation (12) our definition of the centre of mass. As before, this will be done by taking a linear combination of the position vectors of the individual particles, where each vector is weighted by the fraction of the total mass carried by the corresponding particle. Thus in terms of the total mass

$$M = m_1 + m_2 + \cdots + m_n = \sum_{i=1}^{n} m_i,$$

the **centre of mass** of the system has position vector $\mathbf{R}$ given by

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i. \tag{18}$$

This reduces to Equation (12) in the case $n = 2$, as expected.

We are now ready to analyse the motion of the system. It would be possible, as in the earlier case, to start by considering the situation where no external forces are present, but since little additional complication is involved we shall include these forces from the outset. The external force acting on the ith particle is denoted by $\mathbf{F}_i^{\text{ext}}$ $(i = 1, 2, \ldots, n)$.

Each particle has a total of n forces acting on it. For example, Particle 1 experiences the effects of the external force $\mathbf{F}_1^{\text{ext}}$ and of the inter-particle forces $\mathbf{F}_{12}, \mathbf{F}_{13}, \ldots, \mathbf{F}_{1n}$. Application of Newton's second law to each particle in turn thus leads to the following set of equations of motion:

$$\left.\begin{array}{rcl}
m_1\ddot{\mathbf{r}}_1 &=& \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \cdots + \mathbf{F}_{1n} + \mathbf{F}_1^{\text{ext}}, \\
m_2\ddot{\mathbf{r}}_2 &=& \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \cdots + \mathbf{F}_{2n} + \mathbf{F}_2^{\text{ext}}, \\
m_3\ddot{\mathbf{r}}_3 &=& \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \cdots + \mathbf{F}_{3n} + \mathbf{F}_3^{\text{ext}}, \\
m_4\ddot{\mathbf{r}}_4 &=& \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \cdots + \mathbf{F}_{4n} + \mathbf{F}_4^{\text{ext}}, \\
\vdots & & \vdots \\
m_n\ddot{\mathbf{r}}_n &=& \mathbf{F}_{n1} + \mathbf{F}_{n2} + \mathbf{F}_{n3} + \mathbf{F}_{n4} + \cdots + \mathbf{F}_n^{\text{ext}}.
\end{array}\right\} \tag{19}$$

Continuing to follow the previous development for a two-particle system, the next step is to add these equations together. From Equation (18), the sum of the left-hand sides is just $M\ddot{\mathbf{R}}$. The sum of the external forces is $\mathbf{F}^{\text{ext}}$, where

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \cdots + \mathbf{F}_n^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}.$$

As for the inter-particle forces, corresponding to every force $\mathbf{F}_{ij}$ in the sum of the right-hand sides there is also a force $\mathbf{F}_{ji}$. (In fact, if $\mathbf{F}_{ij}$ appears above the 'diagonal of blanks' in Equations (19) then $\mathbf{F}_{ji}$ appears in the symmetrically opposite position below, and vice versa.) Thus the sum of the inter-particle forces is made up entirely of pairs of the form $\mathbf{F}_{ij} + \mathbf{F}_{ji}$, and by Newton's third law (Equation (17)), each such pair is equal to the zero vector. We are left then with the equation

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}, \tag{20}$$

where the total external force $\mathbf{F}^{\text{ext}}$ is as given above. Equation (20) is identical in form to Equation (16) for the two-particle case, but now M, $\mathbf{R}$ and $\mathbf{F}^{\text{ext}}$ are defined in terms of n particles rather than just two. If there are no external forces present then we have $\mathbf{F}^{\text{ext}} = \mathbf{0}$, and Equation (20) reduces as before to $\ddot{\mathbf{R}} = \mathbf{0}$. In this case the centre of mass motion is given by

$$\mathbf{R} = \mathbf{V}_0 t + \mathbf{R}_0,$$

This was derived as Equation (15) for the two-particle case.

where the constant vectors $\mathbf{V}_0$ and $\mathbf{R}_0$ are respectively the velocity and initial position of the centre of mass.

The results of this subsection are summarized below.

Motion of the centre of mass of an n-particle system

For a system of n particles with masses $m_1, m_2, \ldots, m_n$ and respective position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the *centre of mass* has position vector

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i, \tag{18}$$

where

$$M = m_1 + m_2 + \cdots + m_n = \sum_{i=1}^{n} m_i$$

is the *total mass* of the system. If the particles are acted upon by external forces $\mathbf{F}_1^{\text{ext}}, \mathbf{F}_2^{\text{ext}}, \ldots, \mathbf{F}_n^{\text{ext}}$ respectively, and if the inter-particle forces satisfy Newton's third law, then the equation of motion of the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}, \tag{20}$$

where

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \cdots + \mathbf{F}_n^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$$

is the *total external force* acting on the system. In the absence of external forces, Equation (20) reduces to $\ddot{\mathbf{R}} = \mathbf{0}$, indicating a uniform motion for the centre of mass.

Equation (20) is an important result for modelling purposes. It means that the overall motion of a system of any number of particles can be investigated legitimately by considering the total mass to be concentrated in a single particle, which is located at the centre of mass and acted upon by the total external force. As indicated at the beginning of this subsection, the same result applies to the 'continuous-matter' model for a solid body, as can be shown by considering the limiting case as the number of representative particles is increased.

Consequently, problems can often be simplified by considering just the motion of the centre of mass. The common modelling assumption that an object or system of particles can be treated as if it were just one particle is well-founded, provided that the single particle is located at the centre of mass and that only the overall motion of the system is sought.

This provides a justification for the approach adopted in the previous mechanics units.

Often the total mass and total external force for a system are effectively known from the outset. As an example, the shot used in shot-putting may be regarded as a collection of many particles (representing molecules) about whose inter-particle forces nothing is known. Its total mass is usually 7.26 kg, so if gravity is the only significant external force acting then the shot experiences a total force of $7.26 \times 9.81 \simeq 71.2$ N vertically downwards. The corresponding model of a single particle located at the centre of mass of the shot is precisely what was employed in *Unit 15* Section 4.

Although the motion of the centre of mass may be regarded as typifying that of a system as a whole, it provides no information about the individual motions of constituent particles. For applications where only the gross motion of the system is of interest, as with the shot, it suffices to establish how the centre of mass moves, but in other cases finer detail may be required. Thus the individual oscillations of the particles in a two-particle spring system with no external forces are not closely described by the information that the centre of mass of the system moves uniformly. As you saw in Subsection 2.1, the nature of the inter-particle forces must be taken into account in order to determine the amplitude and frequency of these oscillations.

As pointed out in the Introduction, single-particle models are also incapable of modelling rotational motion. Whether or not it is reasonable to ignore rotation depends on the context of the problem. Thus the spin of a shot does not much influence its trajectory, whereas for a table-tennis ball the effect of spin combined with air resistance is marked. As another example, the centre of mass motion alone is sufficient to describe the way in which the Earth orbits the Sun, but it does not assist in explaining the frequency with which day and night occur on the Earth's surface.

A model for rotational motion is developed in *Unit 29*.

If the motion of individual particles within a system is required then, as with the two-particle spring system, the form of the inter-particle forces is needed. The equation of motion must then be solved for each particle.

Exercise 10

A shell in mid-air explodes into a large number of fragments whose mutual interactions obey Newton's third law. Each fragment experiences the force of gravity, but the effect of air resistance may be neglected. Show that, no matter how many fragments there are or how they move individually, the centre of mass $\mathbf{R}$ of the whole collection moves in such a way that

$$\ddot{\mathbf{R}} = -g\mathbf{k},$$

where $\mathbf{k}$ is a unit vector directed vertically upwards.

[*Solution on page 50*]

2.4 Locating the centre of mass

In principle, Equation (20) determines the motion of the centre of mass. When more information is required about the way in which a system of particles or solid object moves, it is natural to provide this by describing what motion takes place relative to the centre of mass, but in order to do so one needs to know the precise position of the centre of mass within the system. This may be found by direct application of the definition (Equation (18)), but in many cases there is an easier approach. We therefore conclude this section with some further remarks about locating the centre of mass.

A solid object can be mentally divided up into many chunks of matter, each of which is represented by a particle. Suppose that the object is sufficiently symmetric to have a point within it which is recognizable as the 'geometric centre'; a sphere and a cylinder are two such examples. Suppose also that the object is *homogeneous*, which means that it is of the same physical consistency throughout. Then the many-particle representation of the object can be chosen so that

(i) each particle has the same mass m, and

(ii) for any one particle of the system with position vector $\mathbf{r}_i$, there is another particle with position vector $-\mathbf{r}_i$.

The second of these conditions implies that the origin O is chosen to be at the geometric centre of the object, as illustrated for the sphere and cylinder in Figure 13 below.

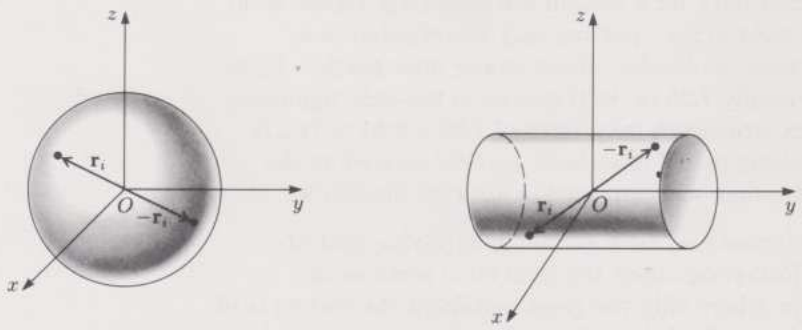


Figure 13

In the summation on the right-hand side of Equation (18), each term $m\mathbf{r}_i$ is now coupled with a term $m(-\mathbf{r}_i)$, so the overall result is the zero vector. The outcome $\mathbf{R} = \mathbf{0}$ shows that the centre of mass lies at the geometric centre. The particular choice of origin here simplified the argument, but the result obtained is independent of it.

Rule 1

The centre of mass of a symmetric homogeneous object is located at the geometric centre of the object.

A similar result applies to any symmetric system of particles where, instead of homogeneity, each particle has the same mass.

Having introduced this rule for symmetric objects, we proceed to state another which applies in different circumstances. Consider a system which consists of n distinct objects, each with its own total mass M_i and centre of mass $\mathbf{R}_i$ $(i = 1, 2, \ldots, n)$. Figure 14 illustrates this situation for the case $n = 3$. It can be shown, by considering each of the constituent objects as represented by a system of particles and then applying Equation (18), that the centre of mass $\mathbf{R}$ of the whole system is given by

$$\mathbf{R} = \frac{1}{M}(M_1\mathbf{R}_1 + M_2\mathbf{R}_2 + \cdots + M_n\mathbf{R}_n) = \frac{1}{M}\sum_{i=1}^{n} M_i\mathbf{R}_i,$$

where $M = M_1 + M_2 + \cdots + M_n$ is the total mass of the system. In words, this rule is as follows.

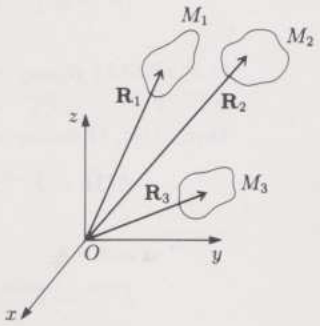


Figure 14

Rule 2

The centre of mass of a collection of objects can be found by treating each object as if it were a single particle situated at its own centre of mass.

A similar result applies to a collection of distinct systems of particles.

Rules 1 and 2 apply respectively to symmetric homogeneous objects and to collections of objects. They may also be used in conjunction to find the centre of mass of any object which can be regarded as a collection of symmetric homogeneous objects. For example, the bolt shown on the left of Figure 15 may be decomposed into the two cylindrical objects shown on the right. If these cylinders are each homogeneous (even though not necessarily both made from the same material) then their individual centres of mass can be found by applying Rule 1. From this information and their total masses, Rule 2 gives the centre of mass of the original object. A similar approach is required in the exercise below, which completes the section.

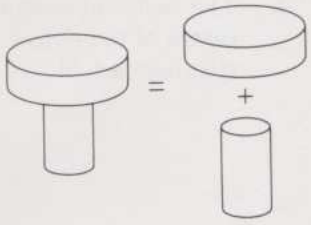


Figure 15

Exercise 11

A more realistic model of the dumb-bell from Exercise 5 consists of two spheres connected by a thin cylindrical rod. The spheres are made of different materials, though each is homogeneous, as is the rod. Each sphere has the same radius. Sphere 1 is centred at the origin and has mass 3 kg. Sphere 2 is centred at the point with position vector $0.3\mathbf{i} + 0.4\mathbf{j}$ and has mass 1 kg. The cylindrical rod has mass 0.1 kg.

What is the position of the centre of mass of the dumb-bell according to this model?

[*Solution on page 50*]

Summary of Section 2

1. **Newton's third law** states that the inter-particle forces acting mutually between a pair of particles are equal in magnitude and opposite in direction. If the particles are labelled i and j, then this law may be expressed in terms of vectors by the equation

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

where $\mathbf{F}_{ij}$ is the force on Particle i due to the effect of Particle j, and vice versa.

2. For a system of n particles with masses $m_1, m_2, \ldots, m_n$ and respective position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the **centre of mass** has position vector

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i,$$

where

$$M = m_1 + m_2 + \cdots + m_n = \sum_{i=1}^{n} m_i$$

is the **total mass** of the system. If the particles are acted upon by external forces $\mathbf{F}_1^{\text{ext}}, \mathbf{F}_2^{\text{ext}}, \ldots, \mathbf{F}_n^{\text{ext}}$ respectively, and if the inter-particle forces satisfy Newton's third law, then the equation of motion of the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}},$$

where

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \cdots + \mathbf{F}_n^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$$

is the **total external force** acting on the system.

3. In the absence of external forces the equation of motion of the centre of mass reduces to $\ddot{\mathbf{R}} = \mathbf{0}$, indicating a uniform motion

$$\mathbf{R}(t) = \mathbf{V}_0 t + \mathbf{R}_0,$$

where the constant vectors $\mathbf{V}_0$ and $\mathbf{R}_0$ are respectively the velocity of the centre of mass and its initial position (at time $t = 0$).

4. The centre of mass of a symmetric homogeneous object is located at the geometric centre of the object. The centre of mass of a collection of objects can be found by treating each object as if it were a single particle situated at its own centre of mass.

3 Linear momentum and energy

In Section 2 we dwelt mainly on the motion of the centre of mass of a many-particle system. Only in the case of the two-particle spring system in Subsection 2.1 were the individual particle motions derived in full, though you saw in Examples 2 and 3 of Section 2, and in Exercises 7 and 9, how the position and velocity of a pair of particles at a given time may be related to the corresponding data at an earlier time. This theme is taken up once more in the current section, where we are concerned with particle velocities rather than with their positions.

For much of the section we shall assume that the inter-particle forces are significant only over a finite time interval, which may be considered as the duration of a *collision* or *explosion* for the system of particles. Both collisions and explosions have the typical property that large, unknown forces act between the particles over a relatively short period of time while the particles are close together. Despite a lack of detailed knowledge about the inter-particle forces, we can relate the state of the system after the interaction has taken place to its state beforehand. As previously, it is assumed that the inter-particle forces satisfy Newton's third law.

The main concepts to be used in these investigations are *linear momentum* and kinetic energy, both of which apply either to a single particle or to a system. The introduction of linear momentum leads to a formulation of Newton's second law which is slightly more general than that used hitherto. As a consequence of this law, the linear momentum of a system is *conserved* (or remains constant) during an interaction with no external forces acting. Under certain circumstances, the kinetic energy of the system may also be conserved.

3.1 Linear momentum and Newton's second law

We start by giving a new name to a product of familiar quantities. The **linear momentum** of a particle is defined to be the product of its mass and its velocity. Thus a particle of mass m with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has linear momentum $\mathbf{p}$, where

The adjective 'linear' is included here to distinguish this type of momentum from the *angular momentum* to be introduced in *Units 28* and *29*.

$$\mathbf{p} = m\dot{\mathbf{r}} = m\dot{x}\mathbf{i} + m\dot{y}\mathbf{j} + m\dot{z}\mathbf{k}. \tag{1}$$

This equation may also be written as $\mathbf{p} = m\mathbf{v}$, where $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity of the particle. Clearly linear momentum is a vector quantity, with its direction being identical to that of the particle's velocity. The magnitude of the linear momentum may be thought of as the particle's 'resistance to being stopped'. To make this more precise, if a particle moving in one dimension has linear momentum $10\,\mathrm{kg\,m\,s^{-1}} = 10\,\mathrm{N\,s}$ in the positive x-direction, then it takes a time of 10 s for the particle to be brought to rest by a constant opposing force of magnitude 1 N. If the particle had twice as much momentum, then it would take twice as long for the same constant opposing force to bring it to rest, and so on.

The SI units for the magnitude of linear momentum are $\mathrm{kg\,m\,s^{-1}}$ (mass times speed), or equivalently N s.

Note that the single linear momentum magnitude of $10\,\mathrm{kg\,m\,s^{-1}}$ covers a range of possible particle states. For example, the particle might have mass 1000 kg and speed $0.01\,\mathrm{m\,s^{-1}}$ ('heavy and slow'), or it might have mass 0.1 kg and speed $100\,\mathrm{m\,s^{-1}}$ ('light and fast').

Exercise 1

A particle of mass 3 kg has position vector

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}.$$

What is the linear momentum of the particle at time $t = 0$?

[*Solution on page 50*]

Differentiation of Equation (1) gives

We assume here that m is constant. This point is discussed further below.

$$\dot{\mathbf{p}} = m\ddot{\mathbf{r}}.$$

Now if $\mathbf{F}$ is the total force acting on the particle then, by Newton's second law, we have

$$m\ddot{\mathbf{r}} = \mathbf{F}, \tag{2}$$

from which it follows that

$$\dot{\mathbf{p}} = \mathbf{F}. \tag{3}$$

This equation states that the rate of change of linear momentum for a particle is equal to the total force acting on it. Since this is a vector equation, it equates each component of $\mathbf{p}$ to the corresponding component of $\mathbf{F}$. For example, consideration of the x-components gives $m\ddot{x} = F_x$, as in *Unit 15* Section 2.

If no net force acts on the particle then $\mathbf{F} = \mathbf{0}$, and Equation (3) becomes

$$\dot{\mathbf{p}} = \mathbf{0},$$

showing that $\mathbf{p}$ is a constant vector throughout the motion. Thus if the total force is zero then the linear momentum of the particle does not vary with time. Any physical quantity that remains constant in time is called a *conserved quantity*. The statement that the linear momentum of a particle is a conserved quantity in the absence of force is often called the *law of conservation of linear momentum* for a single particle, although it is really just another way of expressing Newton's first law.

Having defined the linear momentum $\mathbf{p}$of a particle and expressed Newton's second law in terms of $\mathbf{p}$ (Equation (3)), we next extend the concept of linear momentum to a system of n particles. Suppose, as in Section 2, that the particles of such a system have masses $m_1, m_2, \ldots, m_n$ and respective position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. According to the definition provided by Equation (1), the ith particle of the system has linear momentum $\mathbf{p}_i$, where

$$\mathbf{p}_i = m_i\dot{\mathbf{r}}_i = m_i\mathbf{v}_i \qquad (i = 1, 2, \ldots, n).$$

Here $\mathbf{v}_i = \dot{\mathbf{r}}_i$ is the velocity of the ith particle.

Then the **total linear momentum P** of the system is defined to be the sum of the momenta of the individual particles, that is,

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n = \sum_{i=1}^{n} \mathbf{p}_i. \tag{4}$$

You are asked to apply this definition in the following exercise.

Exercise 2

Figure 1 shows a system of two particles, which have masses m_1, m_2 and respective velocities $\mathbf{v}_1, \mathbf{v}_2$. Suppose that $m_1 = 1$, $m_2 = 2$ and $\mathbf{v}_1 = 3\mathbf{j}$. Find the total linear momentum of the system for each of the following values of $\mathbf{v}_2$.

(i) $\mathbf{v}_2 = \mathbf{i} - 6\mathbf{j}$

(ii) $\mathbf{v}_2 = -1.5\mathbf{j}$

[*Solution on page 51*]

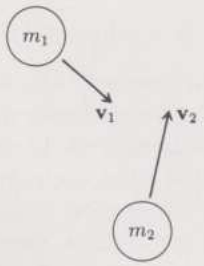


Figure 1

We next express the total linear momentum **P** of the system in terms of the total mass M and centre of mass **R**. From Equation (18) of Section 2, we have

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n).$$

Differentiation gives the velocity $\dot{\mathbf{R}}$ of the centre of mass as

$$\dot{\mathbf{R}} = \frac{1}{M}(m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + \cdots + m_n\dot{\mathbf{r}}_n) = \frac{1}{M}(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n),$$

We assume as before that the mass of each particle is constant.

since $\mathbf{p}_i = m_i\dot{\mathbf{r}}_i$ $(i = 1, 2, \ldots, n)$. It then follows from Equation (4) that

$$\mathbf{P} = M\dot{\mathbf{R}}. \tag{5}$$

From Equation (20) of Section 2, we also have $M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}$, so that differentiation of Equation (5) leads to

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}. \tag{6}$$

Equation (5) states that the total linear momentum of a system is equal to the product of the total mass and the velocity of the centre of mass. In other words, the total linear momentum is the same as that of a single particle placed at the system's centre of mass, where the mass of the particle is the total mass of the system. Equation (6) says that the rate of change of total linear momentum is equal to the total external force. This again tells us that it is valid to deduce the motion of the centre of mass of a system from consideration of the single particle described above, provided that this particle is regarded as being acted upon by the total external force.

In the absence of external forces, Equation (6) reduces to

$$\dot{\mathbf{P}} = \mathbf{0},$$

so that the total linear momentum **P** is a constant vector (is conserved) as in the single-particle case. From Equation (5), this implies as before that the centre of mass moves uniformly.

Exercise 3

A system, which consists of two particles joined by a spring of negligible mass, is launched across a frictionless horizontal surface represented by the (x, y)-plane. Particle 1 has mass 3 kg, and Particle 2 has mass 1 kg. At the time of launch ($t = 0$, say) Particle 1 has velocity $0.75\mathbf{i} + 0.5\mathbf{j}$, while Particle 2 has velocity $0.75\mathbf{i} - 1.5\mathbf{j}$.

(i) Find the total linear momentum of the system at time $t = 0$.

(ii) Write down the relationship to be satisfied by the velocities of the two particles at any later time, assuming that no external forces act on the system.

[*Solution on page 51*]

You will have noticed that the results deduced for a system of particles are remarkably similar in form to those for a single particle. The following table summarizes these similarities, and brings out the direct analogy between the concepts of

(i) mass m, position $\mathbf{r}$, linear momentum $\mathbf{p}$ and total force $\mathbf{F}$ for a single particle, and

(ii) total mass M, position $\mathbf{R}$ of the centre of mass, total linear momentum $\mathbf{P}$ and total external force $\mathbf{F}^{\text{ext}}$ for a many-particle system.

	Single particle	Many-particle system
Linear momentum	$\mathbf{p} = m\dot{\mathbf{r}}$	$\mathbf{P} = M\dot{\mathbf{R}}$
Newton's second law	$m\ddot{\mathbf{r}} = \mathbf{F}$ $\dot{\mathbf{p}} = \mathbf{F}$	$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}$ $\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$
In absence of external force	$m\ddot{\mathbf{r}} = \mathbf{0}$ $\dot{\mathbf{p}} = \mathbf{0}$ $\mathbf{p}$ is constant	$M\ddot{\mathbf{R}} = \mathbf{0}$ $\dot{\mathbf{P}} = \mathbf{0}$ $\mathbf{P}$ is constant

You may be wondering at this stage why we have introduced the concepts of linear momentum $\mathbf{p}$ and total linear momentum $\mathbf{P}$, since so far they have merely permitted us to rewrite Newton's second law, without apparently leading to any new results. One answer is that the introduction of $\mathbf{p}$ and $\mathbf{P}$ provides some notational simplification in the equations of motion. A more important reason is that the equation

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{2}$$

The following discussion applies equally to the corresponding equations of motion for systems of particles.

is less general in its application than is the equation

$$\dot{\mathbf{p}} = \mathbf{F}. \tag{3}$$

In fact, Equation (2) applies only for a particle with *constant* mass, whereas Equation (3) applies even when the mass of the particle is *variable*. To see that at most one of these equations can be valid in the variable-mass case, note that the right-hand side of the definition of linear momentum, $\mathbf{p} = m\dot{\mathbf{r}}$, is a product of two quantities. Consequently, differentiation of this equation gives

$$\dot{\mathbf{p}} = \dot{m}\dot{\mathbf{r}} + m\ddot{\mathbf{r}},$$

showing that the left-hand sides of Equations (2) and (3) are the same only when $\dot{m} = 0$, that is, when the mass of the particle is constant. Although we earlier derived Equation (3) from Equation (2) under the assumption that m was constant, it is actually Equation (3) which is the more basic version of Newton's second law and Equation (2) which is a special case. This fact does not, of course, invalidate any of our previous work, since all the particles considered so far have had constant mass.

The idea of a 'variable-mass particle' may seem strange at first. It arises because a particle may be chosen to model a physical object whose nature changes with time. A relevant example, to be considered in Section 5, concerns the motion of a rocket due to the ejection of exhaust gases from its tail. This situation can be modelled by a system of two particles, one representing the rocket with its unspent fuel and the other representing the exhaust gases. The mass of the 'rocket' particle decreases with time as fuel is spent, and there is a corresponding increase in the mass of the 'gases' particle. The acceleration of the rocket is caused partly by the external forces present (such as gravity) and partly by the 'rate of recoil' from the exhaust gases, demonstrating that Equation (2) does not hold for the 'rocket' particle.

In other words, a particle may be chosen in such a way that it represents different quantities of matter at different times.

The results of this subsection are summarized below.

Linear momentum and Newton's second law

The *linear momentum* $\mathbf{p}$ of a particle with mass m and position vector $\mathbf{r}$ is defined by

$$\mathbf{p} = m\dot{\mathbf{r}}.$$

In terms of linear momentum, *Newton's second law* can be expressed as

$$\dot{\mathbf{p}} = \mathbf{F}, \tag{3}$$

where $\mathbf{F}$is the total force acting upon the particle. In the absence of any force on the particle, its linear momentum is conserved (that is, $\mathbf{p}$ is constant). If the mass m of the particle is constant then Equation (3) reduces to

$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{2}$$

For a many-particle system, the *total linear momentum* $\mathbf{P}$ is the sum of the momenta of the individual particles. It is related to the total mass M of the system and the position of the centre of mass $\mathbf{R}$ by the equation

$$\mathbf{P} = M\dot{\mathbf{R}}. \tag{5}$$

Newton's second law leads to the result

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}, \tag{6}$$

where $\mathbf{F}^{\text{ext}}$ is the total external force acting on the system. In the absence of external forces, the total linear momentum is conserved (that is, $\mathbf{P}$ is constant). If the total mass M of the system is constant then Equation (6) reduces to

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}.$$

Having raised the possibility of a particle with variable mass, we shall not consider such a case further until Section 5. The remainder of this section deals only with constant-mass particles. However, you will see situations where, within a given system, two particles coalesce to form a single particle having the same total mass. This eventuality can be included in the constant-mass context by thinking of the final single particle as being the same as the two initial particles superposed. A similar approach can be applied to a system consisting of one initial particle which 'explodes' into two or more distinct particles.

3.2 Momentum conservation during a collision

This subsection concentrates on applications of Equation (6) in cases where no net external forces act upon the particles of a system. Under these circumstances, as you have seen, Equation (6) takes the form

$$\dot{\mathbf{P}} = \mathbf{0},$$

implying that the total linear momentum $\mathbf{P}$ is conserved during the motion. Consider a system with two particles, which have masses m_1, m_2 and corresponding position vectors $\mathbf{r}_1, \mathbf{r}_2$. For this system, conservation of total linear momentum means that the vector expression

$$\mathbf{P}(t) = m_1\dot{\mathbf{r}}_1(t) + m_2\dot{\mathbf{r}}_2(t) \tag{7}$$

is constant throughout the motion. In other words, the value of the right-hand side is at all times equal to the value which it has at one specific time, $t = 0$ say. Hence if the masses m_1, m_2 and the initial velocity vectors $\dot{\mathbf{r}}_1(0), \dot{\mathbf{r}}_2(0)$ are given, then there is a known relationship between the particle velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ at any subsequent time. You looked at an example of this phenomenon in Exercise 3.

All the systems considered in this subsection are two-particle systems, though the principles may be more widely applied.

Consequently, at any time after $t = 0$, a given value for the velocity of one of the particles permits us to deduce the value of the other particle's velocity at that time. This is, in fact, the same as the process which you carried out as part of Exercise 7 of

Section 2. There, for particles of equal mass, the value of $\dot{\mathbf{r}}_2(2)$ was deduced from given values of $\dot{\mathbf{r}}_1(0)$, $\dot{\mathbf{r}}_2(0)$ and $\dot{\mathbf{r}}_1(2)$.

In what follows we confine attention to situations in which the inter-particle forces between a pair of particles may be ignored outside a certain finite time interval, which represents the time during which the particles are close enough for each to have a significant effect on the motion of the other. As mentioned earlier, the interval of significant interaction may be thought of as modelling a *collision*. Examples of collisions are provided by an impact between snooker balls, magnetic pucks repelling each other and the embedding of a bullet in its target.

The largest inter-particle distance for which the inter-particle force is significant is called the *range* of the force.

Note that, as in the case of the pucks, such a collision need not involve actual physical contact between the objects which the particles represent, since inter-particle forces may act without the mediation of any physical connection. The word 'collision' should therefore be regarded as including an interaction which is just a 'close encounter'.

The restriction to a finite time interval of interaction makes no difference to our treatment of momentum conservation, beyond allowing us the simplification of thinking in terms of 'before' and 'after' the interaction. A more telling reason for making this restriction will be provided in the next subsection.

A collision in the absence of external forces may be modelled as a three-stage process.

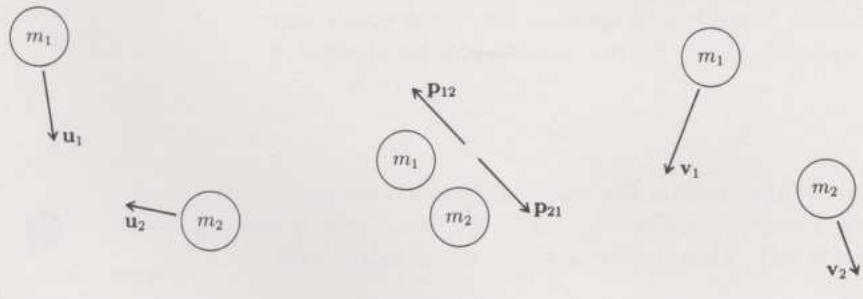
Before: During the first stage the particles are moving with constant velocity, usually towards each other. The inter-particle forces are not yet significant.

Interaction: In the second stage the inter-particle forces are significant, and consequently the velocities of the particles are changing. However, as a consequence of Newton's third law, the total linear momentum of the system remains constant.

After: During the third stage the particles are again moving with constant velocity, usually away from each other. The inter-particle forces are no longer significant.

These three stages are illustrated in Figure 2 below, which also introduces notation for the particle velocities at either end of the collision. As previously, the masses of the two particles are denoted by m_1 and m_2. Their velocities prior to the interaction are $\mathbf{u}_1$ and $\mathbf{u}_2$, and the corresponding velocities following the interaction are $\mathbf{v}_1$ and $\mathbf{v}_2$.

The significance of the vectors $\mathbf{p}_{12}$ and $\mathbf{p}_{21}$ in Figure 2(b) will be explained shortly.



(a) Before interaction (b) Interaction (c) After interaction

Figure 2

We shall use the conservation of total linear momentum to relate the states of the two-particle system before and after the interaction stage. This will enable us to predict the value of one of the outgoing velocities, $\mathbf{v}_2$ say, from knowledge of the two incoming velocities $\mathbf{u}_1, \mathbf{u}_2$ and of the other outgoing velocity $\mathbf{v}_1$. From Equation (7), the total linear momentum before the interaction is

$$\mathbf{P}_{\text{before}} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2,$$

whereas the total linear momentum after the interaction is

$$\mathbf{P}_{\text{after}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2.$$

Since the total linear momentum is conserved throughout the collision, we have

$$\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}},$$

from which

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2. \tag{8}$$

This equation expresses the conservation of total linear momentum for a two-particle system. Notice that it can be written in the alternative form

$$m_1(\mathbf{v}_1 - \mathbf{u}_1) = -m_2(\mathbf{v}_2 - \mathbf{u}_2). \tag{9}$$

The left-hand side here is the linear momentum change for Particle 1 as a result of the whole interaction with Particle 2. If we denote this momentum change by $\mathbf{p}_{12}$, and the corresponding linear momentum change for Particle 2 due to Particle 1 by $\mathbf{p}_{21}$, then Equation (9) can be rewritten in the simple form

$$\mathbf{p}_{12} = -\mathbf{p}_{21}.$$

This shows that, for a two-particle interaction, conservation of total linear momentum is equivalent to the principle that any momentum 'gain' by one of the particles is balanced by an equivalent 'loss' in the momentum of the other. A more precise and familiar way of putting this is that the momentum changes of the two particles caused by the interaction are equal in magnitude but opposite in direction.

The linear momentum change vectors $\mathbf{p}_{12}$ and $\mathbf{p}_{21}$ are shown in Figure 2(b) above. These momentum changes are also called *impulses*.

You will notice that this is similar in form to Newton's third law for a pair of inter-particle forces (Equation (13) of Section 2). In fact, it may be considered as an integral version of Newton's third law.

Conservation of linear momentum during a collision

In the absence of external forces, the total linear momentum $\mathbf{P}$ of a system of particles is conserved. For a collision, momentum conservation implies that $\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$. In the two-particle case, this is equivalent to

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2, \tag{8}$$

where the particles have masses m_1, m_2, initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$ respectively.

We move next to the application of Equation (8), starting with collisions which take place in one dimension. In this case, Equation (8) reduces to a scalar equation.

Example 1

In snooker, all of the balls have the same mass, m say. Consider a one-dimensional collision in which a white ball, travelling initially with speed u, hits a stationary black ball. After the collision the white ball stops dead. Express the subsequent speed v of the black ball in terms of u.

Solution

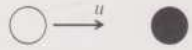
The collision is illustrated in Figure 3. Suppose that the white and black balls are represented respectively by Particles 1 and 2, and that the x-axis is chosen to be in the initial direction of motion of the white ball. Then, in the terminology introduced for Equation (8), we have

$$m_1 = m_2 = m, \quad \mathbf{u}_1 = u\mathbf{i},\ \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = \mathbf{0},\ \mathbf{v}_2 = v\mathbf{i}.$$

Equation (8) therefore gives

$$mu + 0 = 0 + mv,$$

or $u = v$. Hence the velocity of the black ball after the collision is the same as that of the white ball before the collision. □



(a) Before collision

(b) After collision

Figure 3

Exercise 4

In a one-dimensional collision between balls of equal mass m, a white ball with initial speed u hits a stationary black ball. After the collision the two balls move off together with the same speed v. Express this common speed in terms of u.

[*Solution on page 51*]

Example 1 and Exercise 4 demonstrate that, for a two-particle collision with given incoming velocities, there is more than one possible outcome which satisfies conservation of total linear momentum. Additional information about the nature of the collision is required in order to predict a unique outcome. We shall return to this point in Subsection 3.3.

Exercise 5

A bullet travelling at a speed of $1000\,\mathrm{m\,s^{-1}}$ hits a stationary target whose mass is a thousand times as large. Assuming that no external forces act, and that the bullet remains embedded in the target, what is their joint speed after the collision?

[*Solution on page 51*]

In practice, collisions usually involve more than one dimension. The following example and exercise relate to two dimensions. Collisions in three dimensions are treated in a similar way.

Example 2

Two particles of mass 3 kg collide. Before the interaction, Particle 1 has velocity $2\mathbf{i}$ and Particle 2 is stationary. After the interaction, Particle 1 has velocity $\mathbf{i}+\mathbf{j}$.

(i) Find the velocity $\mathbf{v}_2$ of Particle 2 after the interaction.

(ii) Draw a sketch showing the initial and final velocities of both particles.

Solution

(i) According to the data in the question, we have

$$m_1 = m_2 = 3, \quad \mathbf{u}_1 = 2\mathbf{i},\ \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = \mathbf{i}+\mathbf{j}.$$

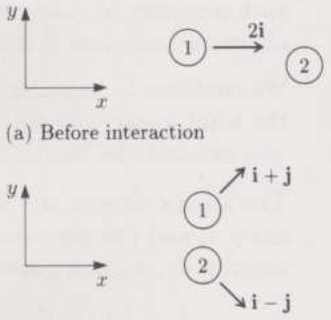
Substituting these values into Equation (8) gives

$$3(2\mathbf{i}) + 3(\mathbf{0}) = 3(\mathbf{i}+\mathbf{j}) + 3\mathbf{v}_2,$$

so that

$$\mathbf{v}_2 = \tfrac{1}{3}(6\mathbf{i} - 3\mathbf{i} - 3\mathbf{j}) = \mathbf{i} - \mathbf{j}.$$

(ii) The sketch is shown in Figure 4. □



(a) Before interaction

(b) After interaction

Figure 4

Exercise 6

Two particles are initially separated by a distance greater than the range of the inter-particle force. Particle 1 (of mass 1 kg) has velocity $2\mathbf{i}+\mathbf{j}$, and Particle 2 (of mass 3 kg) has velocity $\mathbf{j}$. The particles then interact and, as a result, coalesce. They remain stuck together and move as one composite particle (of mass 4 kg). What are the velocity and speed of this composite particle?

[*Solution on page 51*]

As was mentioned earlier, the principle of conservation of total linear momentum may be applied also to an explosion, provided that there are no external forces present. Here the inter-particle forces acting may be very large for a short time, but cease once the explosion has taken place. The following exercises involve explosions, but you can answer them satisfactorily by considering each situation to be a two-particle collision in which the particles are initially coincident.

Exercise 7

A cannon, of mass M, recoils with speed V when a cannon-ball, of mass m, is fired from it with speed v (see Figure 5).

(i) Assuming that the motion is one-dimensional, and that there are no external forces acting, find V in terms of v, M and m.

(ii) If the mass of the cannon is a hundred times that of the cannon-ball, and if the latter is fired at a speed of $100\,\mathrm{m\,s^{-1}}$, with what speed will the cannon recoil?

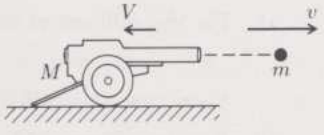
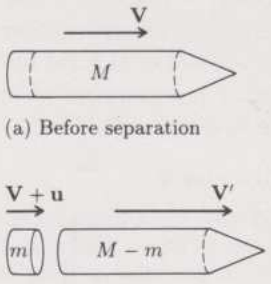


Figure 5

Exercise 8

A rocket, with total mass M and velocity $\mathbf{V}$, jettisons an empty fuel tank of mass m. The tank is jettisoned with velocity $\mathbf{u}$ relative to the rocket, so that its absolute velocity is then $\mathbf{V}+\mathbf{u}$. Assuming that no external forces act, what is the subsequent velocity $\mathbf{V}'$ of the rocket? (Figure 6 illustrates this situation for the one-dimensional case, but the vector notation provides for the general case in which the fuel tank is jettisoned at an arbitrary angle to the rocket's initial direction of motion.)

[*Solutions on page 51*]



(a) Before separation

(b) After separation

Figure 6

The last exercise demonstrates the principle upon which jet engines and rocket motion are based. The backward ejection of exhaust gases causes an increase in speed, while ejection can also be applied sideways for manoeuvring purposes and forwards in order to slow down. However, the expulsion of exhaust gases is a continuous process, whose analysis requires an approach different from that adopted above. We return to this topic in Section 5.

3.3 Energy change during a collision

In each of the two-particle collisions considered in Subsection 3.2, where conservation of total linear momentum was applied, post-collision information about one of the particles had to be supplied in order to predict the subsequent velocity of the other particle. Moreover, Example 1 and Exercise 4 demonstrated that there is more than one possible outcome for a given pair of incoming particle velocities. This subsection explains how such a variety of outcomes can arise, and shows that fewer assumptions need be made about the outcome if more is assumed about the nature of the collision itself.

We continue to consider two-particle systems with no external forces acting, for which the total linear momentum is conserved during a collision. However, we shall now take into account the kinetic energy of the system as well as its linear momentum.

The kinetic energy of a single particle is $\frac{1}{2}\times$ mass $\times$ (speed)^2, which for a particle of mass m and position vector $\mathbf{r}$ is given by $\frac{1}{2}m|\dot{\mathbf{r}}|^2$. For a system of two particles with masses m_1, m_2 and position vectors $\mathbf{r}_1, \mathbf{r}_2$, the total kinetic energy is therefore

$$T = \tfrac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{r}}_2|^2. \tag{10}$$

This expression for the kinetic energy of a particle was used in *Unit 15* Subsection 5.2.

For a collision, where the particles have initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$ respectively, the kinetic energies of the system before and after the interaction are given by

$$T_{\text{before}} = \tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 \quad \text{and} \quad T_{\text{after}} = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2.$$

Unlike the corresponding situation for the total linear momentum $\mathbf{P}$, it is not always the case that $T_{\text{before}} = T_{\text{after}}$. The collision may cause mechanical energy to be transformed into other forms of energy, such as the heat and sound generated by an impact. The kinetic energy of the system is then reduced as a result of the overall collision. For certain interactions, however, little mechanical energy is lost. In these circumstances it is a reasonable modelling approximation to assume that *no* energy is lost. Such a collision, for which

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2, \tag{11}$$

is said to be **elastic**. A collision for which Equation (11) does not hold is said to be **inelastic**.

Exercise 9

(i) For the collision of Example 2, the particle masses and velocities are given by

$$m_1 = m_2 = 3, \quad \mathbf{u}_1 = 2\mathbf{i},\ \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = \mathbf{i}+\mathbf{j},\ \mathbf{v}_2 = \mathbf{i}-\mathbf{j}.$$

Show that this collision is elastic.

(ii) For the collision of Exercise 6, the particle masses and velocities are given by

$$m_1 = 1,\ m_2 = 3, \quad \mathbf{u}_1 = 2\mathbf{i}+\mathbf{j},\ \mathbf{u}_2 = \mathbf{j},\ \mathbf{v}_1 = \mathbf{v}_2 = \tfrac{1}{2}\mathbf{i}+\mathbf{j}.$$

Show that this collision is inelastic, and find how much kinetic energy is lost as a result of the collision.

[*Solution on page 51*]

Note that Equation (11) may hold even though the kinetic energy T from Equation (10) is varying throughout the interaction, since there will normally be transfers between the kinetic energies of the particles and the potential energy associated with the inter-particle forces. This applies only while the inter-particle forces are acting, and any kinetic energy lost in this way will have been regained by the time an elastic collision is complete. Thus it is the total mechanical energy (kinetic plus potential energies) which is conserved during an elastic collision, but in the absence of any force (before and after the interaction) there are no potential energy changes and the kinetic energy itself is constant. These remarks explain why we choose to regard an interaction as taking place over a finite time interval. Without this assumption we would need to take into account the potential energy of the inter-particle forces.

Potential energy and total mechanical energy were introduced in *Unit 7* Section 3.

Elastic collisions

In the absence of external forces, a collision between two particles satisfies the linear momentum conservation equation

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2, \tag{8}$$

where the particles have masses m_1, m_2, initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$ respectively.

If the collision is *elastic* then the kinetic energy equation

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2 \tag{11}$$

is also satisfied. Otherwise the collision is *inelastic*.

We next look at some examples involving elastic collisions, starting with one-dimensional cases. Inelastic collisions will be considered later on. We assume throughout that the objects modelled by the particles have no rotational motion (for instance, balls do not roll or spin). This assumption is necessary because rotation has further kinetic energy associated with it, and we are not in a position to include this energy in the current model.

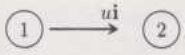
This rotational energy is analysed in *Unit 29*.

Example 3

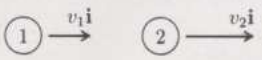
In a one-dimensional snooker collision, where both balls have the same mass, a ball moving with initial speed u hits a stationary ball. Assuming that the collision is elastic, describe its outcome.

Solution

Let m be the mass of each ball, and suppose that the x-axis is chosen in the initial direction of motion of the moving ball. Then the initial particle velocities are respectively $\mathbf{u}_1 = u\mathbf{i}$ and $\mathbf{u}_2 = \mathbf{0}$. Taking the corresponding final velocities to be $\mathbf{v}_1 = v_1\mathbf{i}$ and $\mathbf{v}_2 = v_2\mathbf{i}$, we have the situation shown in Figure 7.



(a) Before collision



(b) After collision

Figure 7

Conservation of total linear momentum (Equation (8)) gives

$$mu + 0 = mv_1 + mv_2$$

or $$u = v_1 + v_2.$$

Conservation of kinetic energy gives

$$\tfrac{1}{2}mu^2 + 0 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2$$

or $$u^2 = v_1^2 + v_2^2.$$

From above, v_2 in this equation may be replaced by $u - v_1$, giving

$$u^2 = v_1^2 + u^2 - 2uv_1 + v_1^2$$

or $$v_1(v_1 - u) = 0.$$

Hence either $v_1 = 0$ or $v_1 = u$. If $v_1 = u$ then $v_2 = 0$, so that $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{u}_2$ and no interaction in fact takes place.

Since it is stated that a collision occurs, we are left with the outcome $v_1 = 0$, for which $v_2 = u$. Hence the collision brings the incoming ball to a standstill, while the ball which was initially stationary moves off with the same velocity as that of the incoming ball. (This outcome is the same as that of Example 1.) □

Exercise 10

What happens in the situation of Example 3 if the balls have different masses, m_1 and m_2 respectively?

[*Solution on page 52*]

Figure 8(a) shows a collision in which the incoming particle velocities $\mathbf{u}_1, \mathbf{u}_2$ are directed parallel to the same straight line. If we specify that this collision is both one-dimensional and elastic, then its unique outcome can be deduced by the approach of Example 3 and Exercise 10, given the values of $\mathbf{u}_1, \mathbf{u}_2$ and of the particle masses m_1, m_2. Note that in describing the collision as one-dimensional we are precluding the possibility that the objects represented by the particles may 'bounce off' each other in some other direction, which might occur if the objects concerned were irregularly shaped, or (in the case of snooker balls) if the impact were a glancing blow rather than a head-on collision. In other words, we are specifying the directions of momentum change for the individual particles to be along the same line as the directions of their initial motions.

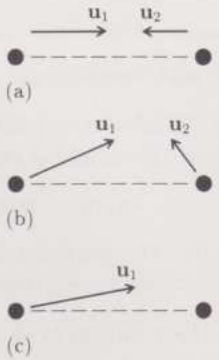


Figure 8

In general, collisions take place in more than one dimension. As pointed out, this could be the case even for the situation in Figure 8(a). It is definitely so for the collision of Figure 8(b), where the incoming velocities are not directed parallel to the same straight line. In Figure 8(c) the collision will probably also be at least two-dimensional.

We shall restrict attention to two-dimensional collisions, but three-dimensional cases can be handled similarly. As with the one-dimensional situation, a unique outcome to a collision can be guaranteed if we specify that the interaction is elastic and identify the line along which the particle momentum changes $\mathbf{p}_{12}$ and $\mathbf{p}_{21}$ are directed. In some cases this line is readily recognizable from the physical situation being modelled. Thus for a collision between snooker balls, it is just the line which joins their centres at the moment of maximum impact.

The linear momentum change vectors for the two particles were introduced following Equation (9) on page 32.

Instead of explicitly stating the line of momentum transfer for the collision, it may be simpler in practice to provide equivalent information, such as the outgoing direction of one of the particles. This is the case with the example and exercise below. For a particle which is initially stationary, the momentum change vector is identical to the outgoing momentum.

Example 4

A white snooker ball, travelling down the centre of the table with speed u, hits a stationary red ball. As a result, the red ball moves off at an angle of 45° to the original direction of motion of the white ball, and is pocketed. If the collision is elastic (and, as usual, the balls have equal mass), find the speeds of the balls after the impact, and decide whether the white ball also enters a pocket.

Solution

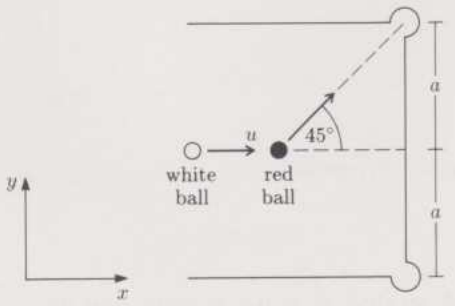


Figure 9

The situation is shown in Figure 9 above, where the x-axis has been chosen to point down the table in the initial direction of motion of the white ball, and the y-axis points

across the table towards the side where the red ball is pocketed. Each ball has mass m. The initial velocities of the two balls are $\mathbf{u}_1 = u\mathbf{i}$ (for the white ball) and $\mathbf{u}_2 = \mathbf{0}$ (for the red ball). Since the red ball moves off at $45°$ to the initial direction of motion of the white ball, its final velocity is of the form $\mathbf{v}_2 = c(\mathbf{i}+\mathbf{j})$, where c is a positive constant. We take the final velocity of the white ball to have components v_x, v_y, so that

$$\mathbf{v}_1 = v_x\mathbf{i} + v_y\mathbf{j}.$$

Conservation of total linear momentum gives

$$mu\mathbf{i} + \mathbf{0} = m(v_x\mathbf{i} + v_y\mathbf{j}) + mc(\mathbf{i}+\mathbf{j}),$$

or, on equating the x-components and the y-components in turn,

$$v_x = u - c, \quad v_y = -c.$$

Conservation of kinetic energy gives

$$\begin{aligned}\tfrac{1}{2}mu^2 &= \tfrac{1}{2}m|v_x\mathbf{i} + v_y\mathbf{j}|^2 + \tfrac{1}{2}m|c(\mathbf{i}+\mathbf{j})|^2\\ &= \tfrac{1}{2}m(v_x^2 + v_y^2) + mc^2.\end{aligned}$$

On substituting the expressions for v_x and v_y obtained above, and multiplying through by $2/m$, this becomes

$$u^2 = (u-c)^2 + c^2 + 2c^2$$

or $$2uc = 4c^2.$$

Since c is non-zero, we then have $c = u/2$, for which

$$v_x = \tfrac{1}{2}u, \quad v_y = -\tfrac{1}{2}u \quad \text{and} \quad \mathbf{v}_1 = \tfrac{1}{2}u(\mathbf{i}-\mathbf{j}).$$

Hence the final speed of each ball is $u/\sqrt{2}$. The white ball moves off at $45°$ to its original direction of motion, and therefore finishes up in the lower pocket shown in Figure 9. □

Exercise 11

Two particles, each of mass m, undergo an elastic collision. Before the interaction, Particle 1 has velocity $3\mathbf{i}$ and Particle 2 is stationary. After the interaction, Particle 1 moves in the direction of the unit vector $\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, where $-\pi/2 < \theta < \pi/2$. Find the final velocity of each particle in terms of the angle θ.

[*Solution on page 52*]

We conclude this section by giving some thought to what happens in inelastic collisions. As a preliminary to this, note that the total kinetic energy of a system of particles cannot be equated with the corresponding quantity for the single representative particle at the centre of mass, as was the case for the total linear momentum. This is confirmed by the result of Exercise 12 below, which gives an alternative expression for the total kinetic energy of a two-particle system.

Exercise 12

(i) Consider a system consisting of two particles which have masses m_1, m_2 and position vectors $\mathbf{r}_1, \mathbf{r}_2$ respectively. Show that the total kinetic energy of the system,

$$T = \tfrac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{r}}_2|^2,$$

can also be written as

$$T = \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}\mu|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1|^2,$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass and

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)$$

is the position vector of the centre of mass. [Hint: If $\mathbf{a}$ is any vector, then $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$.]

The reduced mass of a two-particle system was introduced in Equation (8) of Section 2 (page 12).

(ii) Deduce that, for a collision without external forces in which the particles have initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$, the change in the total kinetic energy as a result of the collision is given by

$$\tfrac{1}{2}\mu(|\mathbf{v}_2 - \mathbf{v}_1|^2 - |\mathbf{u}_2 - \mathbf{u}_1|^2).$$

What can be deduced from this if the collision is elastic?

[*Solution on page 52*]

The result of Exercise 12(i) states that the kinetic energy of a two-particle system is the sum of two parts: one part is the kinetic energy of a representative particle placed at the system's centre of mass, which has mass equal to the total mass of the system, and the other part is internal energy dependent on the relative velocity of the individual particles. In the absence of external forces, the first part of the kinetic energy is constant, so that any change in energy occasioned by a collision is reflected only in the second (internal) part of the energy. In Exercise 12(ii) you then deduced that, for an elastic collision, the initial velocities $\mathbf{u}_1, \mathbf{u}_2$ of the particles and their final velocities $\mathbf{v}_1, \mathbf{v}_2$ are related by the equation

$$|\mathbf{v}_2 - \mathbf{v}_1| = |\mathbf{u}_2 - \mathbf{u}_1|,$$

which states that the relative speed of each particle with respect to the other is unchanged by the collision. For a one-dimensional collision, with

$$\mathbf{u}_1 = u_1\mathbf{i},\ \mathbf{u}_2 = u_2\mathbf{i},\ \mathbf{v}_1 = v_1\mathbf{i},\ \mathbf{v}_2 = v_2\mathbf{i},$$

this leads to the relationship

$$v_2 - v_1 = -(u_2 - u_1), \tag{12}$$

where the extra minus sign is introduced on the right-hand side because a collision reverses the direction of the velocity of each particle relative to the other. This equation holds for a one-dimensional elastic collision, and problems such as Example 3 and Exercise 10 can be solved by applying Equation (12) in place of the equation of kinetic energy conservation (Equation (11)).

The result of Exercise 12(ii) shows also that for an inelastic collision, in which energy is lost and the energy change is therefore negative, we have

$$|\mathbf{v}_2 - \mathbf{v}_1| < |\mathbf{u}_2 - \mathbf{u}_1|.$$

A model for inelastic collisions in one dimension is provided by the formula

$$v_2 - v_1 = -e(u_2 - u_1), \tag{13}$$

which is known as **Newton's law of restitution**. Here e is a constant, called the **coefficient of restitution**, whose value depends on the material of which the colliding objects are made. For an elastic collision we have $e = 1$ (giving Equation (12) once more), while the inequality $e < 1$ holds for an inelastic collision. The value of e indicates how 'bouncy' the objects in the collision are. For example, the coefficients of restitution for glass and for lead are respectively 0.94 and 0.21. It should be noted that Equation (13) is just a mathematical model, which corresponds well with observations in some cases but less closely in others.

From Equation (13) and the result of Exercise 12(ii), the largest possible energy loss occurs when $e = 0$, for which the outcoming particle velocities v_1 and v_2 are equal. Such a collision is said to be *plastic*. You considered plastic collisions in Exercises 4 and 5.

Exercise 13

Example 3 and Exercise 4 featured one-dimensional snooker collisions in which a white ball, travelling with initial velocity $u_1 = u$, hit a stationary black ball ($u_2 = 0$). In Example 3 the collision was assumed to be elastic ($e = 1$), giving $v_1 = 0$ and $v_2 = u$, whereas in Exercise 4 the balls moved off together ($e = 0$) with velocity $v_1 = v_2 = \frac{1}{2}u$.

(i) Verify that the result of Example 3 may be obtained by applying Equation (12) in place of the equation of kinetic energy conservation (Equation (11)).

(ii) Find the outcome of such a collision if the balls are made of material whose coefficient of restitution is $e = 0.8$.

[*Solution on page 52*]

Equation (13) can be written in the alternative form

$$(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{l} = -e(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{l}, \tag{14}$$

where $\mathbf{l}$ is a unit vector in the direction of individual particle momentum change $\mathbf{p}_{12}$ or $\mathbf{p}_{21}$ (that is, $\mathbf{l} = -\mathbf{i}$ or $\mathbf{l} = \mathbf{i}$). The point of this observation, which for the one-dimensional situation is just a complication, is that Equation (14) also expresses Newton's law of restitution for collisions in *more* than one dimension. In words, it says that for a collision between objects with coefficient of restitution e, the component of relative velocity (of either particle with respect to the other) along the direction of

particle momentum transfer is multiplied by the factor $-e$ as a result of the collision. As in the one-dimensional case, we have $e = 1$ for an elastic collision (no energy loss) and $e = 0$ for the maximum energy loss. In the latter case, it follows from Equation (14) that $\mathbf{v}_1 \cdot \mathbf{l} = \mathbf{v}_2 \cdot \mathbf{l}$.

Newton's law of restitution

For a two-particle collision in which the particles have initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$, *Newton's law of restitution* is

$$(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{l} = -e(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{l}, \tag{14}$$

where $\mathbf{l}$ is a unit vector in the direction of individual particle momentum change, and the constant e is the *coefficient of restitution* for the objects represented by the particles. For an elastic collision, the coefficient of restitution is $e = 1$. If the collision is one-dimensional, then Equation (14) reduces to

$$v_2 - v_1 = -e(u_2 - u_1). \tag{13}$$

In order to apply Equation (14), it is necessary to know the direction of momentum change for one of the particles. As pointed out earlier, the momentum change of a particle which is stationary prior to the collision will be identical to its momentum following the collision. This is the situation in the exercise below.

Exercise 14

(i) Verify that the result of Example 4 (where the collision was elastic) may be obtained by applying Equation (14) in place of the equation of conservation of kinetic energy (Equation (11)). (Choose the y-axis to be in the eventual direction of motion of the red ball.)

(ii) Predict the outcome of the collision of Example 4 if it is now assumed that the balls are made from material whose coefficient of restitution is $e = 0.8$.

[*Solution on page 52*]

It was said above that $e \leq 1$ for a collision, with energy loss occurring unless $e = 1$. If an explosion occurs, then extra kinetic energy is introduced into the system from some other source, such as chemical energy. It is then inappropriate to use Newton's law of restitution, since the ratio of outgoing and incoming relative velocity components no longer depends on just the material composition of the colliding objects. However, the result of Exercise 12(ii) still applies, where now the energy change may be positive.

For an explosion, the incoming relative velocity is zero if the particles are initially superposed.

Summary of Section 3

1. The **linear momentum** $\mathbf{p}$ of a particle with mass m and position vector $\mathbf{r}$ is defined by

   $$\mathbf{p} = m\dot{\mathbf{r}}.$$

   In terms of linear momentum, **Newton's second law** can be expressed as

   $$\dot{\mathbf{p}} = \mathbf{F},$$

   where $\mathbf{F}$ is the total force acting upon the particle. In the absence of any force on the particle, its linear momentum is conserved (that is, $\mathbf{p}$ is constant). If the mass m of the particle is constant then Newton's second law reduces to

   $$m\ddot{\mathbf{r}} = \mathbf{F}.$$

2. For a many-particle system, the **total linear momentum** $\mathbf{P}$ is the sum of the momenta of the individual particles. It is related to the total mass M of the system and the position of the centre of mass $\mathbf{R}$ by the equation

   $$\mathbf{P} = M\dot{\mathbf{R}}.$$

Newton's second law leads to the result

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}},$$

where $\mathbf{F}^{\text{ext}}$ is the total external force acting on the system. In the absence of external forces, the total linear momentum is conserved (that is, $\mathbf{P}$ is constant). If the total mass M of the system is constant then the last equation reduces to

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}.$$

3. For a system of particles involved in a collision in the absence of external forces, momentum conservation implies that $\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$. In the two-particle case, this is equivalent to

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2,$$

where the particles have masses m_1, m_2, initial velocities $\mathbf{u}_1, \mathbf{u}_2$ and final velocities $\mathbf{v}_1, \mathbf{v}_2$ respectively.

4. If a two-particle collision is **elastic** then the kinetic energy equation

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2$$

is also satisfied. Otherwise the collision is **inelastic**.

5. **Newton's law of restitution** for a two-particle collision is

$$(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{l} = -e(\mathbf{u}_2 - \mathbf{u}_1)\cdot\mathbf{l},$$

where $\mathbf{l}$ is a unit vector in the direction of individual particle momentum change, and the constant e is the **coefficient of restitution** for the objects represented by the particles. For an elastic collision, the coefficient of restitution is $e = 1$. If the collision is one-dimensional, then this law reduces to

$$v_2 - v_1 = -e(u_2 - u_1).$$

4 Newton's third law (Television Section)

Now watch the television programme.
Read the following after viewing the programme.

TV17

The programme summarized the main ideas of the unit so far, which are consequences of Newton's third law. These ideas were illustrated through relevant experiments with pairs of gliders or pucks modelled by two-particle systems. The programme divided naturally into two parts:

Part 1: motion of the centre of mass of a two-particle system;
Part 2: two-particle collisions.

Part 1 of the programme started by considering a system of two gliders, joined by a light spring and resting on a horizontal air-track. Allan Solomon modelled this apparatus by a two-particle spring system in one dimension, as analysed in Subsection 2.1 (see Figure 1). The equations of motion for the individual particles are

$$m_1\ddot{x}_1 = k(x_2 - x_1 - l_0), \quad m_2\ddot{x}_2 = -k(x_2 - x_1 - l_0),$$

where the spring has stiffness k and natural length l_0. For the centre of mass

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2},$$

the corresponding equation of motion is $\ddot{\bar{x}} = 0$. The resulting prediction that the centre of mass moves with constant velocity was confirmed by observing the motion of the two-glider spring system, both when the gliders were of equal mass and when one glider had three times the mass of the other.

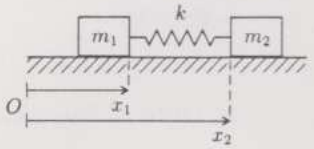


Figure 1

The notation used for the centre of mass in the television programme differs from that elsewhere in the unit. Thus in one dimension the centre of mass is denoted by $\bar{x}$ rather than by X, and in two dimensions it is denoted by $\bar{\mathbf{r}}$ rather than by $\mathbf{R}$.

These results arose from a situation in which no net external forces acted upon the gliders. The track was now tilted, causing the gliders to accelerate down the slope. If g' is the component of gravitational acceleration along the track then, as for Exercise 4 of Section 2, the equation of motion $\ddot{\bar{x}} = g'$ is predicted for the centre of mass. This is the

same as the acceleration which would be experienced by a single glider on the sloping track. Experiment confirmed that the centre of mass of the two-glider spring system had acceleration identical to that of another glider released on a similarly sloping track.

The next experiment dispensed with the spring, examining the interaction between two magnetized gliders on a horizontal air-track. John Bolton demonstrated that the centre of mass of this system moved uniformly (as was the case previously with the spring present). This was interpreted as evidence to support Newton's third law, which predicts uniform motion for the centre of mass. For the current situation, Newton's third law states that the magnetic forces acting mutually between the gliders are equal in magnitude but opposite in direction.

A further experiment concerned the two-dimensional motion of a pair of pucks joined by a spring and moving with minimal friction across a sloping table. The model and notation for this situation are shown in Figure 2, where the inter-particle forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are assumed to satisfy Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$. It was shown that the centre of mass,

$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2},$$

satisfies the equation of motion

$$M\ddot{\bar{\mathbf{r}}} = \mathbf{F}^{\text{ext}},$$

where $M = m_1 + m_2$ is the total mass of the system and $\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}$ is the total external force. The individual external forces are in fact $\mathbf{F}_1^{\text{ext}} = m_1 g'\mathbf{i}$ and $\mathbf{F}_2^{\text{ext}} = m_2 g'\mathbf{i}$, where the unit vector $\mathbf{i}$ is in the downward direction of greatest slope of the table, and g' is the component of gravitational acceleration in this direction. Hence the centre of mass has equation of motion $\ddot{\bar{\mathbf{r}}} = g'\mathbf{i}$, indicating a parabolic trajectory as in Example 3 of Section 2. Experiment confirmed that the centre of mass described a parabola.

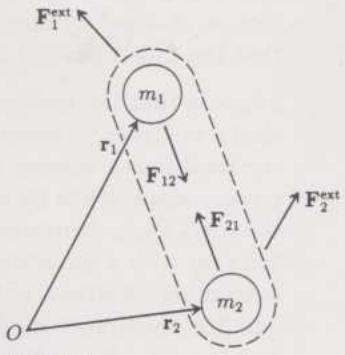


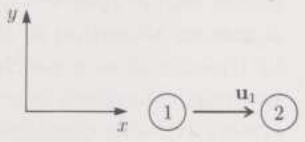
Figure 2

Part 2 of the programme concerned collisions between a pair of magnetized pucks on a horizontal table, whose surface was taken to be the (x, y)-plane. It was assumed, since there was almost no friction between the pucks and the table, that no net external forces acted upon the system. From the resulting equation of motion $\ddot{\bar{\mathbf{r}}} = \mathbf{0}$ for the centre of mass, it was deduced that the total linear momentum $m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$ was constant, and hence that

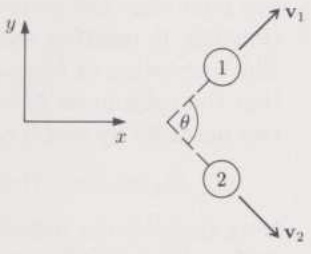
$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2,$$

where $\mathbf{u}_1, \mathbf{u}_2$ are the initial velocities of the particles representing the pucks and $\mathbf{v}_1, \mathbf{v}_2$ are their final velocities. This equation expresses the conservation of total linear momentum for the system.

The first collision, which is illustrated in Figure 3, involved two identical pucks represented by incident and target particles. Initially, the target particle (Particle 2) was stationary at a point on the x-axis, while the incident particle (Particle 1) moved towards the target along a line parallel to the x-axis (so that the collision was slightly off-centre). At a given time after the subsequent collision, both particles were observed to be the same distance away from the x-axis.



(a) Before interaction



(b) After interaction

Figure 3

This result was explained by applying the conservation equation for total linear momentum. Before the interaction, the total linear momentum of the system is $m\mathbf{u}_1 + \mathbf{0} = mu_1\mathbf{i}$, where m is the mass of each particle. This vector has no y-component, so the same must be true for the total linear momentum following the collision, that is, $m(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{j} = 0$. It follows that $\mathbf{v}_1 \cdot \mathbf{j} = -\mathbf{v}_2 \cdot \mathbf{j}$, indicating that the y-components of the final particle velocities differ only in sign and that, at any subsequent time, the particles are at equal distances on either side of the x-axis.

It was further noticed that, following this collision, the particles moved off in directions which were mutually perpendicular (so that $\theta = \pi/2$ in Figure 3(b)). An explanation for this effect was found by assuming the collision to be elastic, in which case the kinetic energy conservation equation

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2$$

applies. You are asked to complete the explanation in the following exercise.

Exercise 1

(i) Consider an elastic collision as described above (in which $\mathbf{u}_2 = \mathbf{0}$), but assume initially that the particles have masses m_1 and m_2 respectively. By applying the equations of conservation for total linear momentum and for kinetic energy, show that the angle θ between the final particle directions of motion is given by

$$\cos\theta = \tfrac{1}{2}(1-\lambda)\frac{|\mathbf{v}_2|}{|\mathbf{v}_1|} \qquad (\mathbf{v}_1 \neq \mathbf{0}),$$

where $\lambda = m_2/m_1$. [Hint: $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos\theta$.]

(ii) Deduce that if the particles have equal mass, then $\theta = \pi/2$.

[*Solution on page 53*]

A similar collision experiment was performed with a pair of non-identical pucks, represented as before by an incident particle and an initially stationary target particle. It was found that the incident particle could now be deflected so that its direction of motion changed by more than $\pi/2$ radians. From this fact it was deduced, as follows, that the target particle must be more massive than the incident particle.

Suppose that the incident particle moves initially as shown in Figure 3(a), and after an elastic collision is deflected as shown in Figure 4. Then its initial momentum, which is the total linear momentum of the system, has a positive x-component and zero y-component, while its final momentum has negative x- and y-components. In order for the total linear momentum to be conserved, the target particle must then leave the collision with a momentum which has positive x- and y-components. Hence the outcoming directions of the incident and target particles are respectively in the third and first quadrants, so that the angle θ between these directions is greater than $\pi/2$ and less than $3\pi/2$. It follows that $\cos\theta < 0$ and, from the result of Exercise 1(i), that $1-\lambda < 0$ or $m_2 > m_1$. In other words, the mass of the target particle must be greater than that of the incident particle.

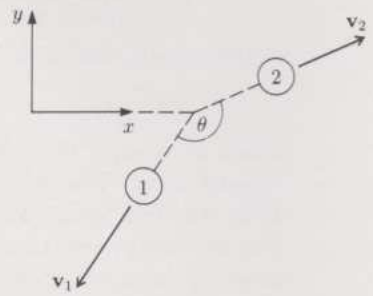


Figure 4

This conclusion is analogous to that made by Rutherford in 1909 when he, together with Geiger and Marsden, fired subatomic particles (called α-particles) at metal foil. They found that some of the α-particles were deflected through angles greater than $\pi/2$ radians, and concluded that the metal atoms must contain particles heavier than the α-particles. This conclusion led to the discovery of the atomic nucleus.

5 Rocket motion

In this short section, we shall derive an equation of motion which models the behaviour of a rocket. The principle behind rocket motion was referred to in Section 3. The rocket engine converts fuel into gas molecules travelling at high speed, whose expulsion causes an alteration in the momentum of the rocket. This occurs continuously, and may be thought of as a controlled explosion. In order to model this process, we consider a two-particle system in which the mass of each particle varies while the total mass of the system remains constant.

To start with, let $t = t_0$ be a particular time at which the rocket engine is running. At this time the system to be considered consists of the rocket and its unspent fuel. At any later time the system is defined to consist of the rocket and unspent fuel (Particle 1) together with any gases expelled from the rocket since $t = t_0$ (Particle 2). The conversion of fuel to exhaust gas is assumed not to involve any loss of matter, so that the total mass M of the system is constant. Denoting the (variable) masses of the two particles by $m_1(t)$ and $m_2(t)$, we have

$$m_1(t) + m_2(t) = M. \tag{1}$$

You may find it easier to visualize the system as being made up of two subsystems of particles rather than two particles. The mathematical argument is the same in either case, since each subsystem may be represented by a particle at its centre of mass.

Note that, by the definitions of the particles, we have $m_1(t_0) = M$ and $m_2(t_0) = 0$. Differentiation of Equation (1) provides $\dot{m}_1 + \dot{m}_2 = 0$, or

$$\dot{m}_2 = -\dot{m}_1, \tag{2}$$

expressing the fact that the rate of mass loss for Particle 1 (the 'rocket' particle) is equal to the rate of mass gain for Particle 2 (the 'gases' particle). The system is illustrated in Figure 1 below, where the velocities of the two particles at time t are

denoted by $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ respectively. Figure 1(a) shows the situation at time $t = t_0$, and Figure 1(b) illustrates the state of the system at some later time.

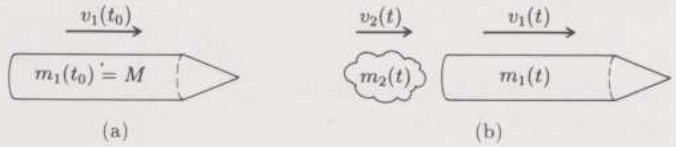


Figure 1

Figure 1(b) corresponds to the case in which the rocket moves in one dimension, with gas being expelled in the direction opposite to the particle's velocity, but the vector notation provides for the more general case in which gas is ejected in any direction.

We now apply Newton's second law in the form

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}},$$

This is Equation (6) of Section 3.

where

$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

is the total momentum of the system, and $\mathbf{F}^{\text{ext}}$ is the total external force. This produces the equation

$$\dot{m}_1\mathbf{v}_1 + m_1\dot{\mathbf{v}}_1 + \dot{m}_2\mathbf{v}_2 + m_2\dot{\mathbf{v}}_2 = \mathbf{F}^{\text{ext}},$$

which from Equation (2) can be rewritten as

$$m_1\dot{\mathbf{v}}_1 - \dot{m}_1(\mathbf{v}_2 - \mathbf{v}_1) + m_2\dot{\mathbf{v}}_2 = \mathbf{F}^{\text{ext}}. \tag{3}$$

Thus far we have a differential equation which holds at or after time $t = t_0$. The next step is to focus on the time $t = t_0$ itself. As observed above, we have $m_2(t_0) = 0$. Putting $\mathbf{u}$ for the velocity of the exhaust gases relative to the rocket at the moment of expulsion, we also have

$$\mathbf{u}(t_0) = \mathbf{v}_2(t_0) - \mathbf{v}_1(t_0).$$

Thus at $t = t_0$, Equation (3) gives

$$m_1(t_0)\dot{\mathbf{v}}_1(t_0) - \dot{m}_1(t_0)\mathbf{u}(t_0) = \mathbf{F}^{\text{ext}}(t_0).$$

Since the 'gases' particle has not yet materialized at time $t = t_0$, the right-hand side of this equation must then be the external force acting on the 'rocket' particle alone. The time $t = t_0$ was chosen arbitrarily, so that the motion of the 'rocket' particle should satisfy the equation at *any* time while its engine is running. Finally, since we no longer have any explicit reference to Particle 2, we can simplify matters by removing the subscript 1 throughout. This leaves the equation of motion known as the **rocket equation**,

$$m\dot{\mathbf{v}} - \dot{m}\mathbf{u} = \mathbf{F}^{\text{ext}}, \tag{4}$$

where the rocket has mass m and velocity $\mathbf{v}$, is acted upon by an external force $\mathbf{F}^{\text{ext}}$ and emits exhaust gases with relative velocity $\mathbf{u}$. This may also be written with $\ddot{\mathbf{r}}$ in place of $\dot{\mathbf{v}}$, where $\mathbf{r}$ is the position vector of the rocket. It can then be seen as analogous to the equation $m\ddot{\mathbf{r}} = \mathbf{F}$ for a constant-mass particle.

Equation (4) should be considered as a differential equation for the unknown velocity function $\mathbf{v}(t)$. In order to solve this equation, it is necessary to have a description of m, $\dot{m}$, $\mathbf{u}$ and $\mathbf{F}^{\text{ext}}$. If all of these quantities are known in terms of t, then the general solution for $\mathbf{v}(t)$ can in principle be obtained by direct integration of the equation

$$\dot{\mathbf{v}} = \frac{1}{m}(\dot{m}\mathbf{u} + \mathbf{F}^{\text{ext}}).$$

In the absence of any external force, this reduces to

$$\dot{\mathbf{v}} = \frac{\dot{m}}{m}\mathbf{u}. \tag{5}$$

For our purposes, it will normally be assumed that in a given situation the velocity $\mathbf{u}$ of the exhaust gases relative to the rocket is constant. Typically, information about the rate of mass loss $-\dot{m}$ will be provided, from which it is possible to obtain an expression for $m(t)$ by solving a preliminary differential equation. This procedure is illustrated by the example after the following summary.

The rate of mass *loss* is $-\dot{m}$ because the rate of mass *change* $\dot{m}$ is negative.

The rocket equation

The velocity $\mathbf{v}$ of a rocket satisfies the *rocket equation*

$$m\dot{\mathbf{v}} - \dot{m}\mathbf{u} = \mathbf{F}^{\text{ext}}, \tag{4}$$

where the rocket (with unspent fuel) has mass m, is acted upon by an external force $\mathbf{F}^{\text{ext}}$ and emits exhaust gases with relative velocity $\mathbf{u}$.

In the absence of any external force, the rocket equation reduces to

$$\dot{\mathbf{v}} = \frac{\dot{m}}{m}\mathbf{u}. \tag{5}$$

Example 1

Suppose that gases are expelled from a rocket with constant rate of mass loss k.

(i) Obtain an expression for the mass $m(t)$ of the rocket and unspent fuel at time t, given that $m(0) = m_0$.

(ii) Find an expression for the velocity $\mathbf{v}(t)$ of the rocket, given that the relative velocity $\mathbf{u}$ of the exhaust gases is constant, that no external forces act and that $\mathbf{v}(0) = \mathbf{v}_0$.

Solution

(i) The rate of mass loss is $-\dot{m} = k$ (constant), so that direct integration gives

$$m = -kt + c,$$

where c is a constant. Since $m = m_0$ when $t = 0$, we have $c = m_0$ and

$$m = m_0 - kt.$$

(This equation holds from $t = 0$ onwards. Note, however, that it can apply for only a finite time thereafter, because fuel is being burnt at a constant rate and will eventually be exhausted. Even if we imagined the entire rocket to be composed of fuel, there would be none left after time $t = m_0/k$, when the equation above gives $m = 0$.)

(ii) With $\mathbf{F}^{\text{ext}} = \mathbf{0}$, we may use the form of the rocket equation given in Equation (5). Putting $m = m_0 - kt$ and $\dot{m} = -k$, we obtain

$$\dot{\mathbf{v}} = \frac{-k}{m_0 - kt}\mathbf{u}.$$

Here $\mathbf{u}$ is a constant vector, so that direct integration produces

$$\mathbf{v} = \log_e(m_0 - kt)\mathbf{u} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. Now $\mathbf{v} = \mathbf{v}_0$ when $t = 0$, giving $\mathbf{c} = \mathbf{v}_0 - \log_e(m_0)\mathbf{u}$ and

$$\mathbf{v} = \mathbf{v}_0 + \log_e\left(\frac{m_0 - kt}{m_0}\right)\mathbf{u}. \quad \square$$

Exercise 1

Repeat Example 1 for the alternative case in which gases are expelled with a rate of mass loss which is proportional to the mass of the rocket and unspent fuel.

[*Solution on page 53*]

A rocket may move in one or more dimensions. In one dimension, the exhaust gases may be ejected either forwards (in the same direction as the velocity of the rocket) or backwards (in the opposite direction to the rocket's velocity). Backward ejection of gases causes forward acceleration of the rocket and occurs, for example, when the rocket is taking off. Forward gas ejection causes deceleration, as required prior to landing or docking with another spacecraft. We shall briefly examine these one-dimensional cases.

Suppose that the rocket is travelling in the direction of the unit vector $\mathbf{i}$ with speed v, and that F^{ext} is the x-component of the external force. Then we have

$$\mathbf{v} = v\mathbf{i} \quad \text{and} \quad \mathbf{F}^{\text{ext}} = F^{\text{ext}}\mathbf{i}.$$

Suppose also that the exhaust gases are ejected with speed u relative to the rocket. If these gases are ejected *forwards* then $\mathbf{u} = u\mathbf{i}$, and Equation (4) becomes

$$m\dot{v} - \dot{m}u = F^{\text{ext}}, \tag{6}$$

whereas if the gases are ejected *backwards*, then $\mathbf{u} = -u\mathbf{i}$ and Equation (4) gives

$$m\dot{v} + \dot{m}u = F^{\text{ext}}. \tag{7}$$

The remaining exercises require the application of Equation (7). Problems calling upon Equation (6) can be treated in a similar way.

Exercise 2

While rising vertically from the surface of the Earth, a rocket of mass m experiences a gravitational force of magnitude mg. Suppose that other external forces (such as air resistance) may be ignored.

(i) Express the velocity v of the rocket in terms of m and the time t, given that the exhaust gases are ejected downwards with constant relative speed u, and that $t = 0$ and $m = m_0$ at the moment of take-off (when $v = 0$).

(ii) Suppose that the rate of mass loss is proportional to the remaining mass of the rocket and unspent fuel. Express v in terms of t for this case.

Exercise 3

While a rocket is still on the launch pad, it experiences a reaction force of magnitude R from the ground, directed vertically upwards. The value of R decreases as the start of the flight approaches, reaching zero at the moment of take-off.

(i) Write down a version of the rocket equation appropriate for the situation while the rocket is on the launch pad, and deduce a condition which must be satisfied by the rate of mass loss $-\dot{m}$ at the moment of take-off.

(ii) Suppose that the rate of mass loss is a constant, k, and that at time $t = 0$ the rocket is on the launch pad with mass $m = m_0$. Express the time of take-off in terms of k, m_0, g and the speed u with which gases are expelled downwards from the rocket.

Exercise 4

A rocket travelling in deep space has no external force acting upon it. At a given moment, when half the mass of the rocket consists of its unspent fuel, it has speed $100\,\text{m}\,\text{s}^{-1}$. Gases are ejected backwards at a constant relative speed of $50\,\text{m}\,\text{s}^{-1}$ until the fuel is completely used up. Calculate the final speed achieved by the rocket.

[*Solutions on page 54*]

Summary of Section 5

1. The velocity $\mathbf{v}$ of a rocket satisfies the **rocket equation**

$$m\dot{\mathbf{v}} - \dot{m}\mathbf{u} = \mathbf{F}^{\text{ext}},$$

where the rocket (with unspent fuel) has mass m, is acted upon by an external force $\mathbf{F}^{\text{ext}}$ and emits exhaust gases with relative velocity $\mathbf{u}$.

2. In the absence of any external force, the rocket equation reduces to

$$\dot{\mathbf{v}} = \frac{\dot{m}}{m}\mathbf{u}.$$

6 End of unit exercises

Section 2

Exercise 1

Two particles move along a frictionless straight horizontal track, attracting one another by inter-particle forces which obey Newton's third law. The first particle has mass 4 kg and experiences no external forces. The second particle has mass 1 kg and is pulled along the track by an external force of magnitude 20 N. It is found that both particles accelerate along the track, staying a fixed distance apart. Find the common acceleration of the particles and the magnitude of the inter-particle forces.

Exercise 2

Two particles of equal mass move in one dimension. They attract one another with inter-particle forces which obey Newton's third law, and they experience no external forces. At time $t = 0$, Particle 1 is at rest at the origin while Particle 2 is at $x = 3$, moving away from the origin with speed $6\,\mathrm{m\,s^{-1}}$. At time $t = 0.5$, Particle 2 has returned to its initial position. Where is Particle 1 at this time?

Exercise 3

Two particles slide along a straight horizontal track. They repel one another by inter-particle forces which obey Newton's third law, and they also experience external forces due to friction. Particle 1 has mass 3 kg and Particle 2 has mass 0.5 kg. The coefficient of kinetic friction between each of the particles and the track is $\mu' = 0.4$. Find the acceleration of the centre of mass when both particles move along the track in the positive x-direction.

[*Solutions on page 54*]

Section 3

Exercise 4

A railway engine of mass M, moving on a straight horizontal track, collides with a stationary truck of mass m. The truck becomes attached to the engine, and both move off together. Express the final speed v of the engine and truck in terms of M, m and the speed u of the engine prior to the collision, assuming that all frictional effects may be neglected.

Exercise 5

A particle of mass m_1, travelling in the positive x-direction, collides with a particle of mass m_2 which was moving previously in the positive y-direction. Both particles have the same speed u prior to the collision. After the collision the particles move off together, with the same velocity. Find their outgoing speed and direction.

Exercise 6

A spacecraft is travelling directly away from the Earth, in a region where the effects of gravity are negligible. Its speed relative to the Earth is $100\,\mathrm{m\,s^{-1}}$, when it launches a satellite in a direction at right angles to its own original line of flight. The satellite, whose mass is a tenth that of the remaining spacecraft, travels subsequently with speed $100\,\mathrm{m\,s^{-1}}$. What effect does the satellite launch have on the motion of the spacecraft?

Exercise 7

A white snooker ball, travelling with speed u, hits a stationary green ball. As a result, the green ball moves off at an angle of $60°$ to the original direction of motion of the white ball. Assuming that the collision is elastic, find the outgoing velocity of each ball. (Take the x-axis in the initial direction of motion of the white ball, and suppose that the subsequent velocity of the green ball has a positive y-component.)

Exercise 8

Prior to a plastic collision (with coefficient of restitution $e = 0$) between two balls of equal mass, the first ball has speed u while the second ball is stationary. The subsequent direction of motion of the second ball is at an angle of $45°$ to the direction of approach of the first ball. Find the outcoming velocities of both balls. (Choose the x-axis to be in the eventual direction of motion of the second ball, and take the initial velocity of the first ball to be $(u/\sqrt{2})(\mathbf{i} - \mathbf{j})$.)

[*Solutions on page 54*]

Section 5

Exercise 9

A rocket moves in one dimension, in the absence of external forces. When its engine is running, it loses mass at a constant rate k and expels exhaust gases with constant relative speed u. The rocket is initially at rest, with mass m_0.

(i) The engine is turned on for a time interval of duration τ, causing the rocket to accelerate. Show that at the end of this interval the rocket has speed

$$v(\tau) = \log_e\left(\frac{m_0}{m_0 - k\tau}\right).$$

(ii) Subsequently, the engine is turned on once again, but with the exhaust gases being expelled in the opposite direction, causing the rocket to decelerate. Find, in terms of k, m_0 and τ, the time which it takes to bring the rocket to rest once more.

[*Solution on page 56*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. (i)

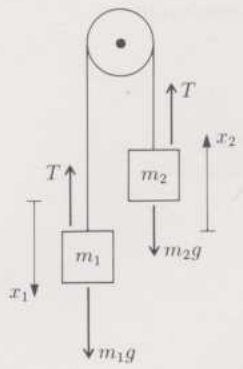


Figure 1

The forces acting on the particles and choice of x-axes are shown above, with the origins for x_1, x_2 being at the initial positions. Application of Newton's second law to each particle gives

$$m_1\ddot{x}_1 = m_1g - T,$$
$$m_2\ddot{x}_2 = T - m_2g.$$

Now $\ddot{x}_1 = \ddot{x}_2$ (since the string is inextensible), so that on adding these equations we have

$$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$$

or $$\ddot{x}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g.$$

The initial conditions are $\dot{x}_1(0) = x_1(0) = 0$, and similarly for x_2, so integrating twice produces

$$x_2 = x_1 = \frac{1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) gt^2.$$

Each particle has constant acceleration, with the heavier particle moving downwards.

(ii)

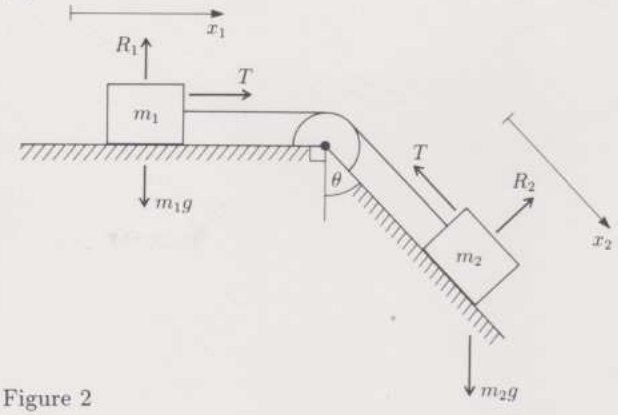


Figure 2

The forces acting on the particles and choice of x-axes are shown above, with the origins for x_1, x_2 again being at the initial positions. Application of Newton's second law to Particle 1 gives, in the horizontal direction,

$$m_1\ddot{x}_1 = T, \qquad (1)$$

whereas for Particle 2, in the direction down the slope, we have

$$m_2\ddot{x}_2 = m_2g\cos\theta - T.$$

Now $\ddot{x}_1 = \ddot{x}_2$, so the sum of these equations is

$$(m_1 + m_2)\ddot{x}_1 = m_2g\cos\theta.$$

This is equivalent to

$$\ddot{x}_1 = \frac{m_2g\cos\theta}{m_1 + m_2}, \qquad (2)$$

from which, integrating twice and applying the initial conditions, we obtain

$$x_2 = x_1 = \frac{1}{2}\left(\frac{m_2g\cos\theta}{m_1 + m_2}\right) t^2.$$

Each particle has constant acceleration, with Particle 2 moving down the sloping plane.

2. The approach required is similar to that in Frame 10 of the audio-tape. The frictional force, of magnitude F say, acts horizontally to the left upon Particle 1. If μ is the coefficient of (static and kinetic) friction then $F \leq \mu R_1$. Now $R_1 = m_1g$, since Particle 1 does not move in the vertical direction, so that

$$F \leq \mu m_1 g.$$

Equation (1) above is modified to

$$m_1\ddot{x}_1 = T - F,$$

following which Equation (2) becomes

$$\ddot{x}_1 = \frac{m_2g\cos\theta - F}{m_1 + m_2}.$$

Motion can occur only if $m_2g\cos\theta$ is greater than the maximum possible value for F, which is $\mu m_1 g$. Thus we have the following two cases.

Case 1: $\mu m_1 \geq m_2\cos\theta$.

In this case $F = T = m_2g\cos\theta$, and the system remains stationary.

Case 2: $\mu m_1 < m_2\cos\theta$.

Here F has its maximum possible value $\mu m_1 g$, so we obtain

$$\ddot{x}_1 = \frac{m_2g\cos\theta - \mu m_1 g}{m_1 + m_2}$$

or $$x_2 = x_1 = \frac{1}{2}\left(\frac{m_2\cos\theta - \mu m_1}{m_1 + m_2}\right) gt^2.$$

The system moves as in the frictionless case, but with a smaller acceleration.

Solutions to the exercises in Section 2

1. (i) It is possible to solve the given differential equation by direct application of Procedures 1.1 and 2.2 from *Unit 6*. However, this process was described in detail for an equation of identical form in *Unit 7*, so it is reasonable to use the result obtained there. Equation (9) of *Unit 7* Section 2 was

$$m\ddot{x} + kx = kl_0.$$

The general solution of this equation was found to be

$$x = l_0 + A\cos(\omega t + \phi),$$

where $\omega = \sqrt{k/m}$ and A, ϕ are arbitrary constants. Hence the given differential equation,

$$\mu\ddot{l} + kl = kl_0,$$

has the general solution

$$l = l_0 + A\cos(\omega t + \phi),$$

where $\omega = \sqrt{k/\mu}$ and A, ϕ are arbitrary constants.

(ii) From Equations (5) of Section 2 we have

$$x_1 = X - \frac{m_2}{M}l \quad \text{and} \quad x_2 = X + \frac{m_1}{M}l,$$

where l is given by the result of part (i) and

$$X = V_0t + X_0$$

from Equation (6) of Section 2. Hence the position functions of the two particles are

$$x_1 = V_0t + X_0 - \frac{m_2}{M}\big(l_0 + A\cos(\omega t + \phi)\big)$$

and $$x_2 = V_0t + X_0 + \frac{m_1}{M}\big(l_0 + A\cos(\omega t + \phi)\big).$$

Thus each particle motion is the superposition of a uniform motion and a simple harmonic oscillation.

2. **(i)** As in Example 1, the total mass of the system is $M = 2m$, the reduced mass is $\mu = \frac{1}{2}m$ and the centre of mass position is given by $X = \frac{1}{2}(x_1 + x_2)$. The initial conditions for x_1 and x_2 are

$$x_1(0) = 0,\ \dot{x}_1(0) = 0 \quad \text{and} \quad x_2(0) = l_0,\ \dot{x}_2(0) = v_0.$$

Thus the initial conditions for $l = x_2 - x_1$ and X are

$$l(0) = l_0,\ \dot{l}(0) = v_0 \quad \text{and} \quad X(0) = \tfrac{1}{2}l_0,\ \dot{X}(0) = \tfrac{1}{2}v_0.$$

From Equation (6) of Section 2 we have $X_0 = X(0)$ and $V_0 = \dot{X}(0)$, so that $X = V_0t + X_0$ gives in this case the centre of mass motion

$$X = \tfrac{1}{2}v_0t + \tfrac{1}{2}l_0.$$

From Equation (11) of Section 2 we obtain

$$l(0) = l_0 + A\cos\phi, \quad \dot{l}(0) = -\omega A\sin\phi,$$

and from the initial conditions for l we can therefore deduce that $\phi = -\frac{1}{2}\pi$ and $A = v_0/\omega$. The particular solution for l is then

$$\begin{aligned} l &= l_0 + \frac{v_0}{\omega}\cos(\omega t - \tfrac{1}{2}\pi) \\ &= l_0 + \frac{v_0}{\omega}\sin\omega t, \end{aligned}$$

where $\omega = \sqrt{k/\mu} = \sqrt{2k/m}$. From Equations (10) of Section 2 the corresponding particular solutions for the individual particles are

$$\begin{aligned} x_1 &= X - \tfrac{1}{2}l \\ &= \tfrac{1}{2}v_0t - \frac{v_0}{2\omega}\sin\omega t \end{aligned}$$

and

$$\begin{aligned} x_2 &= X + \tfrac{1}{2}l \\ &= \tfrac{1}{2}v_0t + l_0 + \frac{v_0}{2\omega}\sin\omega t. \end{aligned}$$

(ii) The given initial situation leads to a valid motion within the model for all $t > 0$ provided that the variable l, which represents the length of the spring, is always positive. From the particular solution for l obtained in part (i), we have $l > 0$ at all times provided that the amplitude v_0/ω is less than the natural length l_0. Since $\omega = \sqrt{2k/m}$, this condition is equivalent to

$$v_0 < l_0\sqrt{2k/m}.$$

3. The modified equations of motion are

$$m_1\ddot{x}_1 = T + F \quad \text{and} \quad m_2\ddot{x}_2 = -T,$$

whose sum is

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = F.$$

Using the definition of the centre of mass X (Equation (4) of Section 2), we have

$$M\ddot{X} = F,$$

where $M = m_1 + m_2$.

4. Application of Newton's second law to each particle in turn gives the equations of motion

$$m_1\ddot{x}_1 = T + m_1g \quad \text{and} \quad m_2\ddot{x}_2 = -T + m_2g,$$

whose sum is

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = (m_1 + m_2)g.$$

Putting $M = m_1 + m_2$ and using the definition of the centre of mass X, this becomes

$$M\ddot{X} = Mg,$$

or simply $\ddot{X} = g$. So regardless of any particle oscillations which might take place within the system, the centre of mass behaves like a particle of mass M falling with constant acceleration g due to gravity.

5. **(i)** The two particle positions are shown on the diagram below. Note that there is no need to draw in the z-axis, since this is a two-dimensional situation (that is, both particles lie in the plane $z = 0$).

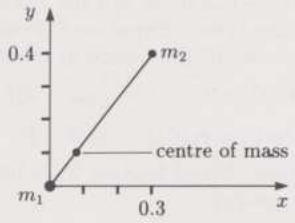


Figure 1

(ii) From the data provided, we have $m_1 = 3$ and $m_2 = 1$, so the total mass of the system is $M = 4$. The two particle position vectors are

$$\mathbf{r}_1 = \mathbf{0} \quad \text{and} \quad \mathbf{r}_2 = 0.3\mathbf{i} + 0.4\mathbf{j}.$$

On substituting these expressions into Equation (12) of Section (2), we obtain

$$\begin{aligned} \mathbf{R} &= \tfrac{1}{4}[3 \times (0\mathbf{i} + 0\mathbf{j}) + 1 \times (0.3\mathbf{i} + 0.4\mathbf{j})] \\ &= 0.075\mathbf{i} + 0.1\mathbf{j}. \end{aligned}$$

(iii) The position of the centre of mass is shown on the diagram in part (i) above. The centre of mass lies on the line joining the particles, dividing the distance between them in the ratio 1:3.

6. **(i)** The inter-particle forces in systems (a) and (d) do not satisfy Newton's third law, since the forces in (a) have the same direction, whereas those in (d) have unequal magnitudes. The forces in system (b) are equal in magnitude and opposite in direction, and therefore satisfy Newton's third law. This is also true of the forces in system (c), even though here the two forces do not act along the line which joins the two particles. However, it is usually the case in models of practical application that the inter-particle forces do act along the line joining the particles. You will see this point discussed further in *Unit 29*.

(ii) On the face of it, the answer here is 'no'. For example, consider the system of Frame 1 on page 6, for which the forces acting on the particles are shown in Frame 2. There is one string force acting on each particle, and while these have the same magnitude T they are clearly not opposite in direction.

However, the answer 'no' applies only if we regard each of the two string forces as being caused directly by the presence of the other particle, ignoring the effect of the pulley. An alternative view is that this system actually comprises three particles, where the third particle represents the pulley and has zero mass. Then this three-particle system *does* satisfy Newton's third law, as illustrated by the full force diagram for the system shown below. The force on the pulley with magnitude F is an external force, being the reaction which the pulley experiences at its mounting.

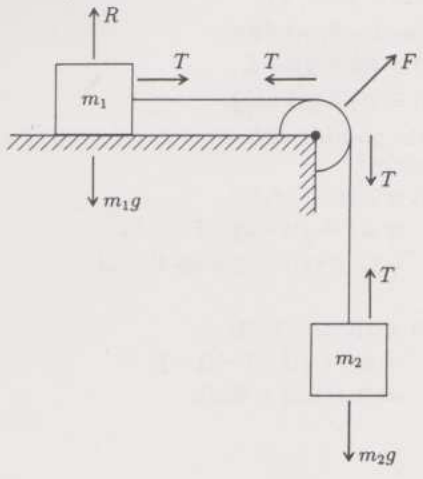


Figure 2

7. If each particle has mass m then the total mass of the system is $M = 2m$. From Equation (12) of Section 2, the position and velocity of the centre of mass at time $t = 0$ are

$$\mathbf{R}(0) = \tfrac{1}{2}\big(\mathbf{r}_1(0) + \mathbf{r}_2(0)\big) = \tfrac{1}{2}(2\mathbf{j} - 2\mathbf{j}) = \mathbf{0},$$

$$\dot{\mathbf{R}}(0) = \tfrac{1}{2}\big(\dot{\mathbf{r}}_1(0) + \dot{\mathbf{r}}_2(0)\big) = \tfrac{1}{2}(4\mathbf{i} - \mathbf{j}).$$

Hence in Equation (15) of Section 2 we have $\mathbf{R}_0 = \mathbf{0}$ and $\mathbf{V}_0 = \frac{1}{2}(4\mathbf{i} - \mathbf{j})$, giving

$$\mathbf{R}(t) = \tfrac{1}{2}(4\mathbf{i} - \mathbf{j})t.$$

In particular, we have

$$\mathbf{R}(2) = 4\mathbf{i} - \mathbf{j} \quad \text{and} \quad \dot{\mathbf{R}}(2) = \tfrac{1}{2}(4\mathbf{i} - \mathbf{j}).$$

However, from the position and velocity given for Particle 1 at time $t = 2$, we also know that

$$\mathbf{R}(2) = \tfrac{1}{2}\big(\mathbf{i} + \mathbf{r}_2(2)\big) \quad \text{and} \quad \dot{\mathbf{R}}(2) = \tfrac{1}{2}\big(\mathbf{i} - \mathbf{j} + \dot{\mathbf{r}}_2(2)\big).$$

On equating the two expressions for $\mathbf{R}(2)$, we obtain

$$\mathbf{r}_2(2) = 2(4\mathbf{i} - \mathbf{j}) - \mathbf{i} = 7\mathbf{i} - 2\mathbf{j}.$$

Further, on equating the two expressions for $\dot{\mathbf{R}}(2)$, we have

$$\dot{\mathbf{r}}_2(2) = 4\mathbf{i} - \mathbf{j} - (\mathbf{i} - \mathbf{j}) = 3\mathbf{i}.$$

Thus at time $t = 2$, Particle 2 is located at the point with coordinates $(7, -2)$ and has velocity $3\,\mathrm{m\,s^{-1}}$ in the positive x-direction.

8. **(i)** Applying Newton's second law to each of the two particles in Figure 10 of Section 2 gives

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{12} + \mathbf{F}_1^{\text{ext}},$$

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{21} + \mathbf{F}_2^{\text{ext}}.$$

(ii) From Newton's third law, we have $\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0}$, so that addition of the two equations in part (i) produces

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}.$$

Using the definition of the position vector $\mathbf{R}$ of the centre of mass (Equation (12) of Section 2), this is equivalent to

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}},$$

where $M = m_1 + m_2$ is the total mass and $\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}$ is the total external force.

9. The situation is described by the same equation of motion as was derived in Example 3 of Section 2, so the general solution for the centre of mass

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$$

is given (as in part (ii) of the solution to that example) by

$$\mathbf{R} = \tfrac{1}{2}gt^2\mathbf{i} + \mathbf{V}_0t + \mathbf{R}_0.$$

As in the solution to Exercise 7, we have $\mathbf{R}_0 = \mathbf{0}$ and $\mathbf{V}_0 = \frac{1}{2}(4\mathbf{i} - \mathbf{j})$, producing

$$\mathbf{R} = (2t + \tfrac{1}{2}gt^2)\mathbf{i} - \tfrac{1}{2}t\mathbf{j}$$

and, on differentiating,

$$\dot{\mathbf{R}} = (2 + gt)\mathbf{i} - \tfrac{1}{2}\mathbf{j}.$$

On putting $t = 2$, we obtain

$$\mathbf{R}(2) = 2(2 + g)\mathbf{i} - \mathbf{j},$$

$$\dot{\mathbf{R}}(2) = 2(1 + g)\mathbf{i} - \tfrac{1}{2}\mathbf{j},$$

so that the position and velocity of Particle 2 at time $t = 2$ are respectively

$$\begin{aligned}\mathbf{r}_2(2) &= 2\mathbf{R}(2) - \mathbf{r}_1(2)\\ &= 4(2 + g)\mathbf{i} - 2\mathbf{j} - \mathbf{i}\\ &= (7 + 4g)\mathbf{i} - 2\mathbf{j} = 46.2\mathbf{i} - 2\mathbf{j}\end{aligned}$$

and

$$\begin{aligned}\dot{\mathbf{r}}_2(2) &= 2\dot{\mathbf{R}}(2) - \dot{\mathbf{r}}_1(2)\\ &= 4(1 + g)\mathbf{i} - \mathbf{j} - (\mathbf{i} - \mathbf{j})\\ &= (3 + 4g)\mathbf{i} = 42.2\mathbf{i}.\end{aligned}$$

10. Suppose that the fragments are represented by n particles, with masses $m_1, m_2, \ldots, m_n$. Then the total mass of the system is

$$M = \sum_{i=1}^{n} m_i,$$

and the external force on the ith particle is $-m_ig\mathbf{k}$, $(i = 1, 2, \ldots, n)$. The total external force is

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{n}(-m_ig\mathbf{k}) = -g\mathbf{k}\sum_{i=1}^{n} m_i = -Mg\mathbf{k}.$$

From Equation (20) of Section 2, the equation of motion of the centre of mass $\mathbf{R}$ is therefore

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}} = -Mg\mathbf{k},$$

which reduces to

$$\ddot{\mathbf{R}} = -g\mathbf{k}.$$

11. The dumb-bell can be regarded as consisting of three parts, namely two spheres and a cylinder. By Rule 1, the two spheres have their centres of mass at $\mathbf{0}$ and $0.3\mathbf{i} + 0.4\mathbf{j}$ respectively. Since both spheres have the same radius, the geometric centre of the connecting cylinder is midway between the centres of the spheres, at $0.15\mathbf{i} + 0.2\mathbf{j}$, and by Rule 1 this is also the centre of mass of the cylinder. The masses and centres of mass of the three component parts are shown on the diagram below.

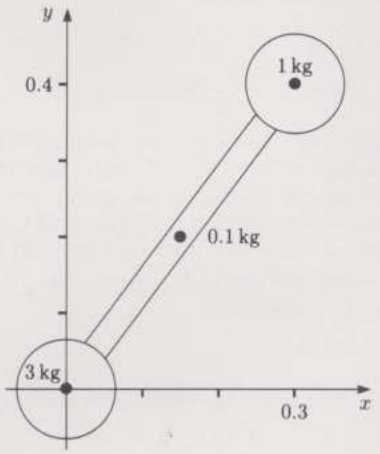


Figure 3

From Rule 2, the centre of mass $\mathbf{R}$ of the model dumb-bell is given by

$$\begin{aligned}\mathbf{R} &= \frac{3 \times \mathbf{0} + 1 \times (0.3\mathbf{i} + 0.4\mathbf{j}) + 0.1 \times (0.15\mathbf{i} + 0.2\mathbf{j})}{(3 + 1 + 0.1)}\\ &= \frac{0.315\mathbf{i} + 0.42\mathbf{j}}{4.1} \simeq 0.077\mathbf{i} + 0.102\mathbf{j}.\end{aligned}$$

This answer is only slightly different from that obtained in the solution to Exercise 5, where it was assumed that the mass of the connecting rod was negligible.

Solutions to the exercises in Section 3

1. We have $m = 3$ and

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}.$$

Differentiation gives

$$\dot{\mathbf{r}}(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j},$$

so that the particle's velocity at time $t = 0$ is

$$\dot{\mathbf{r}}(0) = 2\mathbf{j}.$$

From the definition of linear momentum (Equation (1) of Section 3), we obtain the answer

$$\mathbf{p}(0) = m\dot{\mathbf{r}}(0) = 3 \times 2\mathbf{j} = 6\mathbf{j}.$$

2. In both cases we have

$$m_1 = 1,\ m_2 = 2 \quad \text{and} \quad \mathbf{v}_1 = 3\mathbf{j}.$$

(i) If $\mathbf{v}_2 = \mathbf{i} - 6\mathbf{j}$ then the total linear momentum is

$$\begin{aligned}\mathbf{P} &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2\\ &= 1 \times 3\mathbf{j} + 2 \times (\mathbf{i} - 6\mathbf{j}) = 2\mathbf{i} - 9\mathbf{j}.\end{aligned}$$

(ii) If $\mathbf{v}_2 = -1.5\mathbf{j}$ then the total linear momentum is

$$\begin{aligned}\mathbf{P} &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2\\ &= 1 \times 3\mathbf{j} + 2 \times (-1.5\mathbf{j}) = \mathbf{0}.\end{aligned}$$

(This result demonstrates that the total linear momentum of a system may be zero even though some particles within the system have non-zero momentum.)

3. **(i)** From the given data, the masses of the particles are $m_1 = 3$, $m_2 = 1$, and their initial velocities are

$$\dot{\mathbf{r}}_1(0) = 0.75\mathbf{i} + 0.5\mathbf{j}, \quad \dot{\mathbf{r}}_2(0) = 0.75\mathbf{i} - 1.5\mathbf{j}.$$

Thus the total linear momentum at time $t = 0$ is

$$\begin{aligned}\mathbf{P}(0) &= m_1\dot{\mathbf{r}}_1(0) + m_2\dot{\mathbf{r}}_2(0)\\ &= 3 \times (0.75\mathbf{i} + 0.5\mathbf{j}) + 1 \times (0.75\mathbf{i} - 1.5\mathbf{j}) = 3\mathbf{i}.\end{aligned}$$

(ii) After time $t = 0$ there are no external forces acting on the system, so the total linear momentum is conserved and $\mathbf{P}$ is a constant vector. Hence we have

$$\mathbf{P}(t) = \mathbf{P}(0) = 3\mathbf{i}.$$

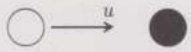
The definition of the total linear momentum at time t gives

$$\begin{aligned}\mathbf{P}(t) &= m_1\dot{\mathbf{r}}_1(t) + m_2\dot{\mathbf{r}}_2(t)\\ &= 3\dot{\mathbf{r}}_1(t) + \dot{\mathbf{r}}_2(t),\end{aligned}$$

so the required relationship between the velocities of the particles is

$$3\dot{\mathbf{r}}_1(t) + \dot{\mathbf{r}}_2(t) = 3\mathbf{i}.$$

4. The collision is as shown in the diagram below.



(a) Before collision

(b) After collision

Figure 1

If Particle 1 represents the white ball and Particle 2 represents the black ball, then we have

$$m_1 = m_2 = m, \quad \mathbf{u}_1 = u\mathbf{i},\ \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = \mathbf{v}_2 = v\mathbf{i},$$

where the x-axis has been chosen in the initial direction of motion of the white ball. Hence, by conservation of total linear momentum (Equation (8) of Section 3), we have

$$mu + 0 = mv + mv,$$

leading to $v = \frac{1}{2}u$. Thus the common velocity of the balls after the collision is equal to half the initial velocity of the incident ball.

5. Suppose that m is the mass of the bullet and that v is the final speed of the bullet and target together. If the bullet and target are represented by Particles 1 and 2 respectively, we have

$$m_1 = m,\ m_2 = 1000m,$$

$$\mathbf{u}_1 = 1000\mathbf{i},\ \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = \mathbf{v}_2 = v\mathbf{i},$$

where the x-axis has been chosen in the initial direction of travel of the bullet. Conservation of total linear momentum gives

$$1000m + 0 = mv + 1000mv,$$

leading to

$$v = \tfrac{1000}{1001} \simeq 1\,\mathrm{m\,s^{-1}}.$$

The target will move at about $1\,\mathrm{m\,s^{-1}}$ after the bullet hits it.

6. From the data given in the question, we have

$$m_1 = 1,\ m_2 = 3, \quad \mathbf{u}_1 = 2\mathbf{i} + \mathbf{j},\ \mathbf{u}_2 = \mathbf{j}.$$

The final velocities of both particles are the same, so we set

$$\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}.$$

Then the conservation of total linear momentum gives

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v},$$

so that the velocity of the composite particle is

$$\begin{aligned}\mathbf{v} &= \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2}\\ &= \frac{1 \times (2\mathbf{i} + \mathbf{j}) + 3 \times \mathbf{j}}{1 + 3} = \tfrac{1}{2}\mathbf{i} + \mathbf{j}.\end{aligned}$$

The speed of the composite particle is

$$|\mathbf{v}| = \sqrt{(\tfrac{1}{2})^2 + 1^2} = \tfrac{1}{2}\sqrt{5} \simeq 1.12\,\mathrm{m\,s^{-1}}.$$

7. **(i)** Suppose that the cannon and ball are represented by Particles 1 and 2 respectively, and that the x-axis is in the direction of the cannon-ball's motion. Then we have

$$m_1 = M,\ m_2 = m, \quad \mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0},\ \mathbf{v}_1 = -V\mathbf{i},\ \mathbf{v}_2 = v\mathbf{i}.$$

Conservation of total linear momentum gives

$$0 = M(-V) + mv,$$

so that

$$V = mv/M.$$

(ii) If $M = 100m$ and $v = 100$ then, from the result of part (i), we have $V = 1$. Hence the recoil speed of the cannon is $1\,\mathrm{m\,s^{-1}}$.

8. After the fuel tank has been jettisoned, the residual mass of the rocket is $M - m$. Conservation of total linear momentum means that

$$M\mathbf{V} = m(\mathbf{V} + \mathbf{u}) + (M - m)\mathbf{V}',$$

from which the final velocity of the rocket is given by

$$\mathbf{V}' = \frac{M\mathbf{V} - m(\mathbf{V} + \mathbf{u})}{M - m} = \mathbf{V} - \frac{m}{M - m}\mathbf{u}.$$

9. **(i)** For $m_1 = m_2 = 3$, $\mathbf{u}_1 = 2\mathbf{i}$, $\mathbf{u}_2 = \mathbf{0}$, $\mathbf{v}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{v}_2 = \mathbf{i} - \mathbf{j}$, we have

$$\begin{aligned}T_{\text{before}} &= \tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2\\ &= \tfrac{1}{2}(3 \times 4 + 3 \times 0) = 6\end{aligned}$$

and

$$\begin{aligned}T_{\text{after}} &= \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2\\ &= \tfrac{1}{2}(3 \times 2 + 3 \times 2) = 6.\end{aligned}$$

So Equation (11) of Section 3 is satisfied in this case, and the collision is elastic.

(ii) For $m_1 = 1$, $m_2 = 3$, $\mathbf{u}_1 = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u}_2 = \mathbf{j}$ and $\mathbf{v}_1 = \mathbf{v}_2 = \frac{1}{2}\mathbf{i} + \mathbf{j}$, we have

$$\begin{aligned}T_{\text{before}} &= \tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2\\ &= \tfrac{1}{2}(1 \times 5 + 3 \times 1) = 4\end{aligned}$$

and

$$\begin{aligned}T_{\text{after}} &= \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2\\ &= \tfrac{1}{2}(1 \times 1.25 + 3 \times 1.25) = 2.5.\end{aligned}$$

Hence Equation (11) of Section 3 is not satisfied in this case, and the collision is inelastic. The loss in kinetic energy as a result of the collision is

$$T_{\text{before}} - T_{\text{after}} = 4 - 2.5 = 1.5\,\mathrm{J}.$$

10. If the balls have respective masses m_1 and m_2 then, with the other data of Example 3, conservation of total linear momentum gives

$$m_1u + 0 = m_1v_1 + m_2v_2, \tag{1}$$

whereas conservation of kinetic energy produces

$$\tfrac{1}{2}m_1u^2 + 0 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2. \tag{2}$$

From Equation (1), we have

$$v_2 = \frac{m_1}{m_2}(u - v_1). \tag{3}$$

On substituting this into Equation (2), we obtain

$$\tfrac{1}{2}m_1u^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2\left(\frac{m_1}{m_2}\right)^2(u^2 - 2uv_1 + v_1^2).$$

After multiplying through by $2m_2/m_1$ and rearranging, this becomes

$$(m_1 + m_2)v_1^2 - 2m_1uv_1 + (m_1 - m_2)u^2 = 0,$$

which is a quadratic equation for v_1 with roots

$$\begin{aligned} v_1 &= \frac{2m_1u \pm \sqrt{4m_1^2u^2 - 4(m_1 + m_2)(m_1 - m_2)u^2}}{2(m_1 + m_2)} \\ &= \frac{m_1 \pm m_2}{m_1 + m_2}u \\ &= u \quad \text{or} \quad \frac{m_1 - m_2}{m_1 + m_2}u. \end{aligned}$$

As in Example 3 we discount the possibility $v_1 = u$, since this corresponds to an absence of any collision. From Equation (3), the remaining outcome is

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u, \quad v_2 = \frac{2m_1}{m_1 + m_2}u.$$

(As a check, note that this reduces to the outcome for Example 3 ($v_1 = 0$, $v_2 = u$) when $m_1 = m_2$.) The answer obtained for v_1 is positive for $m_1 > m_2$ and negative for $m_1 < m_2$, predicting that the incoming ball will continue in the same direction of motion if it is the heavier of the two balls, but reverse its direction if it is the lighter ball.

11. From the data given,

$$m_1 = m_2 = m, \quad \mathbf{u}_1 = 3\mathbf{i}, \ \mathbf{u}_2 = \mathbf{0}.$$

The final velocities of the particles are $\mathbf{v}_1$ and $\mathbf{v}_2$, where

$$\mathbf{v}_1 = |\mathbf{v}_1|\cos\theta\,\mathbf{i} + |\mathbf{v}_1|\sin\theta\,\mathbf{j}.$$

Conservation of total linear momentum gives

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2,$$

that is, on dividing through by m,

$$3\mathbf{i} = |\mathbf{v}_1|\cos\theta\,\mathbf{i} + |\mathbf{v}_1|\sin\theta\,\mathbf{j} + \mathbf{v}_2$$

$$\text{or} \quad \mathbf{v}_2 = (3 - |\mathbf{v}_1|\cos\theta)\mathbf{i} - |\mathbf{v}_1|\sin\theta\,\mathbf{j}. \tag{4}$$

The collision is elastic, so that

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 + \tfrac{1}{2}m_2|\mathbf{u}_2|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2$$

or, on multiplying through by $2/m$,

$$9 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2.$$

On substituting for $\mathbf{v}_2$ from Equation (4), we obtain

$$\begin{aligned} 9 &= |\mathbf{v}_1|^2 + (3 - |\mathbf{v}_1|\cos\theta)^2 + (|\mathbf{v}_1|\sin\theta)^2 \\ &= 9 - 6|\mathbf{v}_1|\cos\theta + 2|\mathbf{v}_1|^2, \end{aligned}$$

from which

$$2|\mathbf{v}_1|(|\mathbf{v}_1| - 3\cos\theta) = 0.$$

Since Particle 1 is moving after the collision, we have $|\mathbf{v}_1| \neq 0$ and so

$$|\mathbf{v}_1| = 3\cos\theta,$$

leading to

$$\mathbf{v}_1 = 3\cos\theta(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$

Also, substitution of the expression for $|\mathbf{v}_1|$ into Equation (4) provides

$$\begin{aligned} \mathbf{v}_2 &= (3 - 3\cos^2\theta)\mathbf{i} - 3\cos\theta\sin\theta\,\mathbf{j} \\ &= 3\sin\theta(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}). \end{aligned}$$

12. (i) Using the hint given and the definition of $\mathbf{R}$, we have

$$\begin{aligned} |\dot{\mathbf{R}}|^2 &= \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} \\ &= \frac{1}{M^2}(m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2) \cdot (m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2) \\ &= \frac{1}{M^2}(m_1^2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 + 2m_1m_2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + m_2^2\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2) \\ &= \frac{1}{M^2}(m_1^2|\dot{\mathbf{r}}_1|^2 + 2m_1m_2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + m_2^2|\dot{\mathbf{r}}_2|^2). \end{aligned}$$

Similarly, we obtain

$$|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1|^2 = |\dot{\mathbf{r}}_1|^2 - 2\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2 + |\dot{\mathbf{r}}_2|^2.$$

These results are now combined in such a way as to remove the $\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2$ terms. Thus, noting that $\mu = m_1m_2/M$, we have

$$\begin{aligned} &\tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}\mu|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1|^2 \\ &= \frac{1}{2M}[(m_1^2 + m_1m_2)|\dot{\mathbf{r}}_1|^2 + (m_2^2 + m_1m_2)|\dot{\mathbf{r}}_2|^2] \\ &= \tfrac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{r}}_2|^2. \end{aligned}$$

This establishes the result that the total kinetic energy T of the system can be written in the alternative form

$$T = \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}\mu|\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1|^2.$$

(ii) Before the interaction, we have $\dot{\mathbf{r}}_1 = \mathbf{u}_1$, $\dot{\mathbf{r}}_2 = \mathbf{u}_2$ and

$$T_{\text{before}} = \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}\mu|\mathbf{u}_2 - \mathbf{u}_1|^2,$$

whereas after the interaction, we have $\dot{\mathbf{r}}_1 = \mathbf{v}_1$, $\dot{\mathbf{r}}_2 = \mathbf{v}_2$ and

$$T_{\text{after}} = \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}\mu|\mathbf{v}_2 - \mathbf{v}_1|^2.$$

Now by the conservation of linear momentum, $\dot{\mathbf{R}}$ is constant throughout the interaction. It follows that the change in kinetic energy as a result of the collision is given by

$$T_{\text{after}} - T_{\text{before}} = \tfrac{1}{2}\mu(|\mathbf{v}_2 - \mathbf{v}_1|^2 - |\mathbf{u}_2 - \mathbf{u}_1|^2).$$

If the collision is elastic then no energy is lost, so that

$$|\mathbf{v}_2 - \mathbf{v}_1| = |\mathbf{u}_2 - \mathbf{u}_1|.$$

13. (i) The initial velocities of the particles are $u_1 = u$ and $u_2 = 0$. By conservation of total linear momentum we have, as in Example 3,

$$u = v_1 + v_2.$$

Equation (12) of Section 3 gives

$$v_2 - v_1 = -(u_2 - u_1) = u.$$

The solution of this pair of simultaneous equations for v_1, v_2 is

$$v_1 = 0, \quad v_2 = u,$$

as was found in the solution to Example 3.

(ii) As in part (i), conservation of total linear momentum gives

$$u = v_1 + v_2.$$

With $e = 0.8$, Equation (13) of Section 3 produces

$$v_2 - v_1 = -0.8(u_2 - u_1) = 0.8u.$$

The solution of this pair of simultaneous equations for v_1, v_2 is

$$v_1 = 0.1u, \quad v_2 = 0.9u.$$

Thus the ball which is initially stationary moves off with $\frac{9}{10}$ the original speed of the incoming ball, whose own speed is reduced to $\frac{1}{10}$ of its previous value.

14. (i)

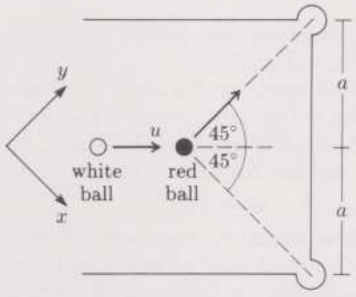


Figure 2

The diagram above is identical to that used in solving Example 4, except that the axes are now chosen with the y-axis pointing in the final direction of motion of the red ball. In terms of these axes, we have

$$\mathbf{u}_1 = \frac{u}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \quad (\text{since } |\mathbf{u}_1| = u)$$

and $\mathbf{u}_2 = \mathbf{0}, \quad \mathbf{v}_2 = c\mathbf{j}$,

where c is a positive constant. Putting

$$\mathbf{v}_1 = v_x\mathbf{i} + v_y\mathbf{j},$$

conservation of total linear momentum gives

$$\frac{mu}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) + \mathbf{0} = m(v_x\mathbf{i} + v_y\mathbf{j}) + mc\mathbf{j}$$

or, in terms of components,

$$\frac{u}{\sqrt{2}} = v_x, \quad \frac{u}{\sqrt{2}} = v_y + c. \tag{5}$$

Since the collision is elastic ($e = 1$) and the momentum change $\mathbf{p}_{21}$ is in the direction of $\mathbf{j}$, Equation (14) of Section 3 takes the form

$$(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{j} = -(\mathbf{u}_2 - \mathbf{u}_1)\cdot\mathbf{j},$$

giving

$$(c\mathbf{j} - v_x\mathbf{i} - v_y\mathbf{j})\cdot\mathbf{j} = \frac{u}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\cdot\mathbf{j}$$

or $$c - v_y = \frac{u}{\sqrt{2}}. \tag{6}$$

The solution of Equations (5) and (6) is

$$v_x = \frac{u}{\sqrt{2}}, \quad v_y = 0, \quad c = \frac{u}{\sqrt{2}}.$$

Allowing for the different choice of axes, this is the same outcome for the vectors $\mathbf{v}_1, \mathbf{v}_2$ as in Example 4. Each ball has outcoming speed $u/\sqrt{2}$, and the white ball travels towards the lower pocket.

(ii) The notation and diagram are as in part (i). As before, conservation of total linear momentum gives Equation (5). With $e = 0.8$, Equation (14) of Section 3 becomes

$$(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{j} = -0.8(\mathbf{u}_2 - \mathbf{u}_1)\cdot\mathbf{j},$$

that is,

$$c - v_y = \frac{0.8u}{\sqrt{2}}. \tag{7}$$

The solution of Equations (5) and (7) is

$$v_x = \frac{u}{\sqrt{2}}, \quad v_y = \frac{0.1u}{\sqrt{2}}, \quad c = \frac{0.9u}{\sqrt{2}}.$$

Thus the final speed of the red ball is $\frac{9}{10}$ of its value in the elastic case, while the final speed of the white ball is slightly greater than in the elastic case. The white ball travels at an angle $\arctan(0.1) \simeq 6°$ to the x-axis, and will therefore fail to fall into the pocket unless the latter is abnormally large.

Solution to the exercise in Section 4

1. (i) Since we have $\mathbf{u}_2 = \mathbf{0}$, conservation of total linear momentum gives

$$m_1\mathbf{u}_1 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2, \tag{1}$$

and conservation of kinetic energy gives

$$\tfrac{1}{2}m_1|\mathbf{u}_1|^2 = \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2. \tag{2}$$

Taking the square of both sides of Equation (1) and then dividing through by $2m_1$ yields

$$\begin{aligned}\tfrac{1}{2}m_1|\mathbf{u}_1|^2 &= \frac{1}{2m_1}|m_1\mathbf{v}_1 + m_2\mathbf{v}_2|^2 \\ &= \frac{1}{2m_1}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)\cdot(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) \\ &= \frac{1}{2m_1}(m_1^2|\mathbf{v}_1|^2 + 2m_1m_2\mathbf{v}_1\cdot\mathbf{v}_2 + m_2^2|\mathbf{v}_2|^2) \\ &= \tfrac{1}{2}m_1|\mathbf{v}_1|^2 + m_2\mathbf{v}_1\cdot\mathbf{v}_2 + \tfrac{1}{2}m_2\lambda|\mathbf{v}_2|^2,\end{aligned}$$

where $\lambda = m_2/m_1$. Subtracting this from Equation (2) leads to

$$\tfrac{1}{2}m_2(1-\lambda)|\mathbf{v}_2|^2 = m_2\mathbf{v}_1\cdot\mathbf{v}_2.$$

Using the definition of the scalar product,

$$\mathbf{v}_1\cdot\mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos\theta,$$

we then obtain

$$\cos\theta = \tfrac{1}{2}(1-\lambda)\frac{|\mathbf{v}_2|}{|\mathbf{v}_1|},$$

provided that $\mathbf{v}_1 \neq \mathbf{0}$.

(ii) If $m_1 = m_2$ then $\lambda = 1$ and the result of part (i) gives $\cos\theta = 0$. It follows that $\theta = \pi/2$. (Elastic collisions between particles of equal mass were considered in Example 4 and Exercises 10(i) and 12 from Section 3. You can check that the outcoming velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular in each case.)

Solutions to the exercises in Section 5

1. (i) The rate of mass loss is $-\dot{m} = \alpha m$, where α is a positive constant (recalling that $\dot{m}$ is negative). The differential equation

$$\dot{m} = -\alpha m$$

may be solved by separation of variables (see Exercise 4(i) in *Unit 2* Section 3) to give the general solution

$$m = ce^{-\alpha t},$$

where c is a constant. Now $m = m_0$ when $t = 0$, so that $c = m_0$ and

$$m = m_0e^{-\alpha t}.$$

(ii) With $\mathbf{F}^{\text{ext}} = \mathbf{0}$, we may use the form of the rocket equation given in Equation (5) of Section 5. Putting $\dot{m} = -\alpha m$, we obtain

$$\dot{\mathbf{v}} = -\alpha\mathbf{u}.$$

(Note that in this particular case the expression for $m(t)$ from part (i) is not required in order to solve the rocket equation.) Since $\mathbf{u}$ is a constant vector, direct integration produces

$$\mathbf{v} = -\alpha t\mathbf{u} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. The initial condition states that $\mathbf{v} = \mathbf{v}_0$ when $t = 0$, giving $\mathbf{c} = \mathbf{v}_0$ and

$$\mathbf{v} = \mathbf{v}_0 - \alpha t\mathbf{u}.$$

2. **(i)** Choosing the unit vector $\mathbf{i}$ in the direction of the rocket's velocity (vertically upwards), the external force is

$$\mathbf{F}^{\text{ext}} = -mg\mathbf{i}.$$

Exhaust gases are ejected in the backwards direction, so Equation (7) of Section 5 applies, giving

$$m\dot{v} + \dot{m}u = -mg$$

or $$\dot{v} = -\frac{u\dot{m}}{m} - g.$$

Now u is a constant, so that integration produces

$$v = -u\log_e m - gt + c,$$

where c is a constant. From the initial conditions, we have $v = 0$ and $m = m_0$ when $t = 0$, giving $c = u\log_e m_0$ and

$$v = u\log_e\left(\frac{m_0}{m}\right) - gt.$$

(ii) The rate of mass loss is $-\dot{m} = \alpha m$ (where α is a positive constant), and the corresponding initial condition is $m(0) = m_0$. Hence, from Exercise 1(i), we have $m = m_0 e^{-\alpha t}$. It follows from part (i) that

$$v = (u\alpha - g)t.$$

(Note that we must have $u\alpha > g$ here, since otherwise the rocket will not rise.)

3. **(i)** The external forces experienced by the rocket on the launch pad are of magnitude mg downwards (due to gravity) and of magnitude R upwards (due to the reaction from the ground). From Equation (7) of Section 5, we then have

$$m\dot{v} + \dot{m}u = R - mg.$$

For take-off, we require $\dot{v} > 0$ when $R = 0$, that is,

$$-\dot{m}u - mg > 0.$$

At the moment of take-off, the rate of mass loss $-\dot{m}$ must therefore satisfy the inequality $-\dot{m} > mg/u$.

(ii) It is given that $-\dot{m} = k$ (constant) and that $m = m_0$ when $t = 0$, so from Example 1(i) we have

$$m = m_0 - kt.$$

The condition obtained in part (i) becomes

$$k > \frac{(m_0 - kt)g}{u},$$

which leads to

$$t > \frac{m_0}{k} - \frac{u}{g}.$$

The rocket is accelerating upwards at any time when this inequality is satisfied, so the moment of take-off is at $t = m_0/k - u/g$, provided that this expression is non-negative. (If the expression is negative, then the thrust of the exhaust gases is sufficient to provide immediate lift-off at $t = 0$.)

4. With $F^{\text{ext}} = 0$, Equation (7) of Section (5) is

$$m\dot{v} + \dot{m}u = 0.$$

Since u is constant, integration gives

$$v = -u\log_e m + c,$$

where c is a constant. From the information given in the question, we have $u = 50$. Suppose that m_0 is the mass of the rocket when $v = 100$. Then we obtain $c = 100 + 50\log_e m_0$ and

$$v = 100 + 50\log_e\left(\frac{m_0}{m}\right).$$

The fuel is completely used up when $m = \frac{1}{2}m_0$, at which time the speed of the rocket is

$$v = 100 + 50\log_e 2 \simeq 135\,\text{m s}^{-1}.$$

Solutions to the exercises in Section 6

1. **(i)** Suppose that the positive x-axis is chosen in the direction of the external force, and that the inter-particle forces have x-components F_{12} and F_{21}. Since the acceleration of the particles is the same, we may put $\ddot{x}$ in place of both $\ddot{x}_1$ and $\ddot{x}_2$. With $m_1 = 4$, $m_2 = 1$, $F_1^{\text{ext}} = 0$, $F_2^{\text{ext}} = 20$, Newton's second law for each of the particles gives

$$4\ddot{x} = F_{12}, \quad \ddot{x} = 20 + F_{21}. \tag{1}$$

On adding these equations together and using Newton's third law, we have $5\ddot{x} = 20$ (which could also have been written down directly as the equation of motion for the centre of mass), or $\ddot{x} = 4$. The common acceleration of the particles is therefore $4\,\text{m s}^{-2}$ in the same direction as the external force. It follows from either of Equations (1) that the magnitude of the inter-particle forces is 16 N.

2. If the position functions of the particles are denoted by $x_1(t), x_2(t)$ respectively, then we have $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 3$, $\dot{x}_2(0) = 6$. The particles have equal mass, so the centre of mass is given by

$$X = \tfrac{1}{2}(x_1 + x_2).$$

From above, the initial conditions for X are $X(0) = 1.5$, $\dot{X}(0) = 3$. In the absence of external forces, we may apply Equation (6) of Section 2,

$$X = V_0 t + X_0,$$

where $X_0 = X(0)$, $V_0 = \dot{X}(0)$. Thus we obtain

$$X = 3t + 1.5.$$

When $t = 0.5$ this gives $X(0.5) = 3$, so that

$$x_1(0.5) = 2X(0.5) - x_2(0.5) = 6 - 3 = 3.$$

Hence when $t = 0.5$, Particle 1 is at $x = 3$, coincident with Particle 2.

3. According to the model for friction given in *Unit 15* Subsection 3.2, the frictional forces have respective x-components

$$F_1^{\text{ext}} = -\mu' m_1 g = -0.4 \times 3g = -1.2g,$$

$$F_2^{\text{ext}} = -\mu' m_2 g = -0.4 \times 0.5g = -0.2g,$$

so that the total external force is

$$F^{\text{ext}} = F_1^{\text{ext}} + F_2^{\text{ext}} = -1.4g.$$

With $m_1 = 3$ and $m_2 = 0.5$, the total mass is $M = 3.5$. The centre of mass X has equation of motion $M\ddot{X} = F^{\text{ext}}$, that is,

$$\ddot{X} = -\frac{1.4g}{3.5} = -0.4g \simeq -3.92.$$

So the acceleration of the centre of mass is $3.92\,\text{m s}^{-2}$ in the negative x-direction.

4. Since there is no friction, the total linear momentum of the system is conserved. Hence we have

$$Mu + 0 = Mv + mv = (M + m)v,$$

from which the final speed of the engine and truck is $v = Mu/(M + m)$.

5. Prior to the collision, the total linear momentum of the system is

$$\mathbf{P} = m_1 u\mathbf{i} + m_2 u\mathbf{j}.$$

If the outgoing particles have mutual speed v and direction given by the angle θ (measured anticlockwise from the positive x-axis), then the total linear momentum after the collision is

$$\mathbf{P} = (m_1 + m_2)v(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$

Conservation of total linear momentum is expressed by the appearance of the same vector quantity $\mathbf{P}$ on the left-hand sides of the two equations above. On equating the

magnitudes of the right-hand sides, we find that the speed of the outgoing particles is

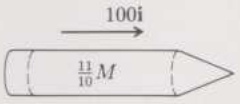
$$v = \frac{u\sqrt{m_1^2 + m_2^2}}{m_1 + m_2}.$$

Equating the ratios of the y- and x-components gives the result

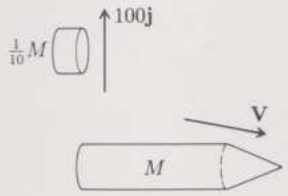
$$\theta = \arctan(m_2/m_1)$$

for the direction of the outgoing particles.

6. If the final mass of the spacecraft is M, then the mass of the satellite is $\frac{1}{10}M$ and the initial mass of the spacecraft plus satellite is $\frac{11}{10}M$. Suppose that the x-axis is chosen in the initial direction of motion of the spacecraft, and that the y-axis is in the launch direction of the satellite. Then the initial velocity of the spacecraft is $100\mathbf{i}$ and the velocity of the launched satellite is $100\mathbf{j}$. Denoting the final velocity of the spacecraft by $\mathbf{V}$, we have the situation shown in the figure below.



(a) Before separation



(b) After separation

Figure 1

Applying conservation of total linear momentum, we obtain

$$\tfrac{11}{10}M \times 100\mathbf{i} = \tfrac{1}{10}M \times 100\mathbf{j} + M\mathbf{V},$$

from which

$$\mathbf{V} = 10(11\mathbf{i} - \mathbf{j}).$$

Hence the speed of the spacecraft after the satellite launch is

$$10\sqrt{11^2 + 1^2} = 10\sqrt{122} \simeq 110.5\,\mathrm{m\,s^{-1}},$$

while the launch causes the spacecraft's direction of motion to deviate by an angle

$$\arctan\tfrac{1}{11} \simeq 5^\circ$$

from its original line of flight.

7. Each ball has mass m. Their initial velocities are $\mathbf{u}_1 = u\mathbf{i}$ (for the white ball) and $\mathbf{u}_2 = \mathbf{0}$ (for the green ball). Since the green ball moves off at 60° to the initial direction of motion of the white ball, its final velocity is of the form

$$\mathbf{v}_2 = c(\cos 60^\circ\mathbf{i} + \sin 60^\circ\mathbf{j}) = c(\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\sqrt{3}\mathbf{j}),$$

where c is a positive constant. We take the final velocity of the white ball to have components v_x, v_y, so that

$$\mathbf{v}_1 = v_x\mathbf{i} + v_y\mathbf{j}.$$

Conservation of total linear momentum gives

$$mu\mathbf{i} + \mathbf{0} = m(v_x\mathbf{i} + v_y\mathbf{j}) + mc(\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\sqrt{3}\mathbf{j}),$$

or, on equating the x-components and the y-components in turn,

$$v_x = u - \tfrac{1}{2}c, \quad v_y = -\tfrac{1}{2}\sqrt{3}c.$$

Conservation of kinetic energy gives

$$\begin{aligned}\tfrac{1}{2}mu^2 &= \tfrac{1}{2}m|v_x\mathbf{i} + v_y\mathbf{j}|^2 + \tfrac{1}{2}m|c(\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\sqrt{3}\mathbf{j})|^2 \\ &= \tfrac{1}{2}m(v_x^2 + v_y^2) + \tfrac{1}{2}mc^2.\end{aligned}$$

On substituting the expressions for v_x and v_y obtained above, and multiplying through by $2/m$, this becomes

$$u^2 = (u - \tfrac{1}{2}c)^2 + (\tfrac{1}{2}\sqrt{3}c)^2 + c^2$$

or $uc = 2c^2$.

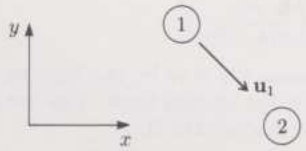
Since c is non-zero, we then have $c = u/2$, for which

$$v_x = \tfrac{3}{4}u, \quad v_y = -\tfrac{1}{4}\sqrt{3}u$$

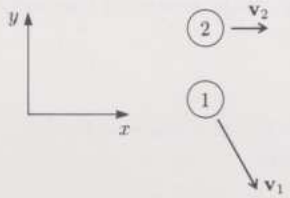
and $\mathbf{v}_1 = \tfrac{1}{2}\sqrt{3}u(\tfrac{1}{2}\sqrt{3}\mathbf{i} - \tfrac{1}{2}\mathbf{j}) = \tfrac{1}{2}\sqrt{3}u(\cos 30^\circ\mathbf{i} - \sin 30^\circ\mathbf{j})$.

Hence the green ball moves off with speed $u/2$ in the direction at 60° to the positive x-axis (measured anticlockwise), while the white ball moves off with speed $\frac{1}{2}\sqrt{3}u$ in the direction at 30° to the positive x-axis (measured clockwise). (Note that the outgoing velocities are at right angles, as predicted for such a collision by the result of Exercise 1(ii) in Section 4.)

8.



(a) Before interaction



(b) After interaction

Figure 2

The interaction is illustrated in the diagram above. Each ball has mass m. We have

$$\mathbf{u}_1 = \frac{u}{\sqrt{2}}(\mathbf{i} - \mathbf{j}), \quad \mathbf{u}_2 = \mathbf{0}, \quad \mathbf{v}_2 = c\mathbf{i},$$

where c is a positive constant. Putting

$$\mathbf{v}_1 = v_x\mathbf{i} + v_y\mathbf{j},$$

conservation of total linear momentum gives

$$\frac{mu}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) + \mathbf{0} = m(v_x\mathbf{i} + v_y\mathbf{j}) + mc\mathbf{i}$$

or, in terms of components,

$$\frac{u}{\sqrt{2}} = v_x + c, \quad -\frac{u}{\sqrt{2}} = v_y. \tag{2}$$

Since the collision is plastic ($e = 0$) and the momentum change $\mathbf{p}_{21}$ is in the direction of $\mathbf{i}$, Equation (14) of Section 3 takes the form

$$(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{i} = 0,$$

giving

$$(c\mathbf{i} - v_x\mathbf{i} - v_y\mathbf{j}) \cdot \mathbf{i} = 0$$

or $c - v_x = 0.$ (3)

The solution of Equations (2) and (3) is

$$v_x = \frac{u}{2\sqrt{2}}, \quad v_y = -\frac{u}{\sqrt{2}}, \quad c = \frac{u}{2\sqrt{2}},$$

giving the final velocities of the balls as

$$\mathbf{v}_1 = \frac{u}{2\sqrt{2}}(\mathbf{i} - 2\mathbf{j}), \quad \mathbf{v}_2 = \frac{u}{2\sqrt{2}}\mathbf{i}.$$

9. **(i)** When the engine is first turned on, the exhaust gases are expelled in the direction opposite to the rocket's velocity. Hence Equation (7) of Section 5 applies, giving (since $F^{\text{ext}} = 0$)

$$\dot{v} = -\frac{\dot{m}}{m}u.$$

The rate of mass loss is $-\dot{m} = k$, so that if $t = 0$ when the engine is turned on, we have $m = m_0 - kt$ (as in Example 1(i) of Section 5). Integration of the rocket equation then produces

$$v = -\log_e(m_0 - kt) + c,$$

where c is a constant. The initial condition for v is $v(0) = 0$, so that $c = \log_e m_0$ and

$$v = \log_e\left(\frac{m_0}{m_0 - kt}\right).$$

Thus when the engine is turned off at time $t = \tau$, the rocket has achieved the speed

$$v(\tau) = \log_e\left(\frac{m_0}{m_0 - k\tau}\right).$$

(ii) We now choose $t = 0$ to be the time at which the engine is switched on for the second time. The initial conditions for this phase of the motion are then

$$m(0) = m_0 - k\tau, \quad v(0) = \log_e\left(\frac{m_0}{m_0 - k\tau}\right).$$

We proceed as in part (i), but using Equation (6) of Section 5 since the exhaust gases are now expelled in the forwards direction. This version of the rocket equation is

$$\dot{v} = \frac{\dot{m}}{m}u,$$

where

$$m = (m_0 - k\tau) - kt = m_0 - k(\tau + t).$$

Integration provides

$$v = \log_e\big(m_0 - k(\tau + t)\big) + c,$$

where c is a constant. The initial condition for v gives

$$\begin{aligned} c &= \log_e\left(\frac{m_0}{m_0 - k\tau}\right) - \log_e(m_0 - k\tau) \\ &= \log_e\left(\frac{m_0}{(m_0 - k\tau)^2}\right), \end{aligned}$$

so that the particular solution for v is

$$v = \log_e\left(\frac{m_0\,(m_0 - k(\tau + t))}{(m_0 - k\tau)^2}\right).$$

The rocket is at rest once more when $v = 0$, that is, when

$$m_0\big(m_0 - k(\tau + t)\big) = (m_0 - k\tau)^2.$$

Solving this equation for t gives

$$t = \tau\left(1 - \frac{k\tau}{m_0}\right),$$

which is the time that it takes to bring the rocket to rest. (Note that this is a shorter time than that for which the engine was first turned on, because the rocket has become lighter.)

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